Quiz 11 Solutions

1. If the curve C is parameterized by $\langle t^2 - t, 2t + 4\rangle$, then compute:

   (a) $ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \sqrt{(2t-1)^2 + 4}\, dt$

   (b) $d\vec{r} = \vec{r}'(t)\, dt = \langle 2t - 1, 2\rangle\, dt$

   (c) Set up the arc length integral for $0 \le t \le 1$.

$$\int_0^1 \sqrt{(2t-1)^2 + 4}\, dt$$

2. If the surface S is parameterized by $\langle x, y, 3x^2 - xy + 5\rangle$ then compute:

   (a) $\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1\rangle = \langle -6x + y, x, 1\rangle$

   (b) $d\vec{S} = \langle -6x + y, x, 1\rangle\, dA$ and $dS = \sqrt{(y - 6x)^2 + x^2 + 1}\, dA$

   (c) Set up the integral for the surface area over the rectangle $0 \le x \le 3, -1 \le y \le 2$

$$\iint_D dS = \int_0^3 \int_{-1}^2 \sqrt{(y-6x)^2 + x^2 + 1}\, dA$$

   (d) The surface normal, $\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} = \frac{1}{\sqrt{(y-6x)^2+x^2+1}}\langle -6x + y, x, 1\rangle$

   (e) If the vector field $\vec{F} = \langle x, y, z^2\rangle$, set up the integral: $\iint_S \vec{F} \cdot d\vec{S}$

$$\iint x(-6x + y) + y^2 + (3x^2 - xy + 5)^2\, dA$$

3. Is $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_C f(x, y)\, ds$? Explain.

   SOLUTION: No. On the LHS of the equation, we are computing the work of a vector field over a curve C, while on the right, we are computing the line integral of a scalar function f on the curve C with respect to arc length.

4. Is $\displaystyle\iint_D g(x, y, z)\, dS = \iint_S \vec{F} \cdot d\vec{S}$? Explain.

   SOLUTION: No. Similar to the equation above, on the right, we are integrating a scalar function g over a surface S with respect to area (typo: The D under the integral should be S- Good for you if you caught it!), but on the left, we are computing the flux of a vector field $\vec{F}$ through a surface S.

5. Is $\int_C \vec{F} \cdot d\vec{r} = \int_C P\, dx + Q\, dy$?

   SOLUTION: Yes, these are the same. Take $\vec{F} = \langle P, Q\rangle$ and $d\vec{r} = \langle dx, dy\rangle$

6. Is $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$?

   SOLUTION: Yes, these are the same. Take

$$d\vec{S} = (\vec{r}_x \times \vec{r}_y) dA = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} |\vec{r}_x \times \vec{r}_y|\, dA = \vec{n}\, dS$$

7. Given surface S over domain D, is $\iint_S \vec{F} \cdot dS = \iint_D \vec{F} \cdot (\vec{r}_x \times \vec{r}_y)\, dA$?

   SOLUTION: Almost the same. The LHS is a vector dotted with a scalar, though (so that notation is meaningless!). To be equal, we would have needed $\vec{F} \cdot d\vec{S}$ on the left.

8. Set up the integral (DO NOT EVALUATE) representing the flux of $\vec{F}$ across the surface S, if the orientation is upward, and

$$\vec{F} = \langle y, x, z^2 \rangle \qquad z = 4 - x^2 - y^2 \qquad 0 \le x \le 1, 0 \le y \le 1$$

   SOLUTION: The normal is already pointing outward and upward:

$$\vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1 \rangle$$

   Now:

$$\int_0^1 \int_0^1 2xy + 2xy + (4 - x^2 - y^2)^2\, dy\, dx$$

9. Set up the integral for the surface area of $4x - 2y + 2z = 4$ above the unit circle in the plane.

$$\vec{r}_x \times \vec{r}_y = \langle 2, -1, 1 \rangle \quad \Rightarrow \quad \sqrt{6} \int_0^{2\pi} \int_0^1 r\, dr\, d\theta$$

10. Set up the integral $\iint_S y\, dS$, if the surface is given by the part of the cone $x^2 + y^2 = z^2$ that lies between the planes $z = 1$ and $z = 3$.

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1^2} = \sqrt{2}$$

   By geometry,

$$\iint_S y\, dS = \sqrt{2} \iint_D y\, dA = 0$$

   If you don't see that, set it up. It cries out to be evaluated:

$$\sqrt{2} \iint_D y\, dA = \sqrt{2} \int_0^{2\pi} \int_1^3 r \sin(\theta)\, r\, dr\, d\theta = \sqrt{2} \int_0^{2\pi} \sin(\theta)\, d\theta \int_1^3 r^2\, dr = 0$$