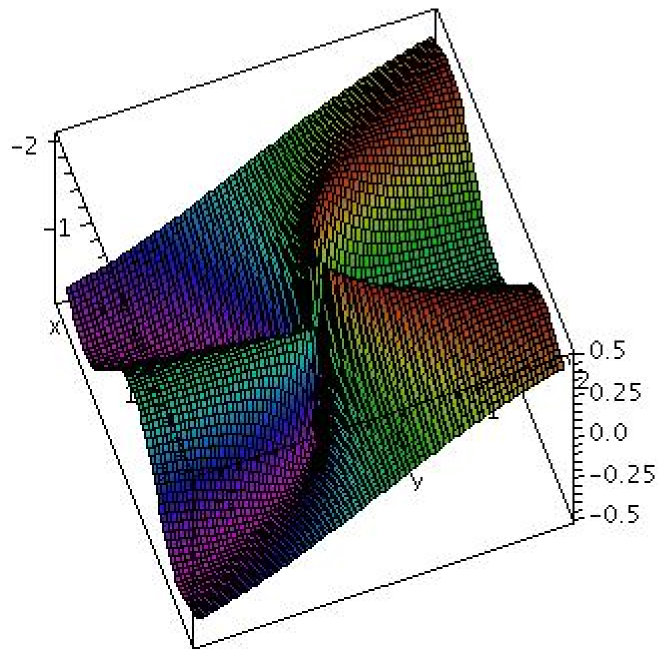
EXAMPLE

In this example, we will see that it is possible for the directional derivative to exist in all possible directions, but still the original function does not have to be differentiable (or even continuous!). The function of interest is:

$$f(x,y) = \begin{cases} x^2y/(x^4+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$



Here is a series of short exercises (solved) that will lead us through the discussion:

1. Show that f is not continuous at the origin by showing that the limit does not exist.

   SOLUTION: For this function, if we move to the origin along any line $y = mx$, then

$$\lim_{(x,mx)\to(0,0)} \frac{x^2mx}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx^3}{x^2(x^2+m^2)} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = \frac{0}{m^2} = 0$$

But, along the curve $y = x^2$, we have:

$$\lim_{(x,x^2)\to(0,0)} \frac{x^2 x^2}{x^4 + x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Conclusion: The limit does not exist at the origin, so this function is not continuous at the origin (and so it is also not differentiable at the origin).

2. Use the definition of the directional derivative to show that the directional derivative exists for all directions $\mathbf{u} = \langle u_1, u_2 \rangle$.

$$\begin{aligned} D_{\boldsymbol{u}} f(0,0) &= \lim_{h\to 0} \frac{f(0 + hu_1, 0 + hu_2) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h^2 u_1^2 h u_2}{h^4 u_1^4 + h^2 u_2^2} - 0}{h} = \lim_{h\to 0} \frac{h^3 u_1^2 u_2}{h \cdot h^2 (h^2 u_1^4 + u_2^2)} \\ &= \lim_{h\to 0} \frac{u_1^2 u_2}{h^2 u_1^4 + u_2^2} \end{aligned}$$

I need to pause here for a moment. What happens in the direction $\mathbf{u} = \langle 1, 0 \rangle$ (or more directly, if $u_2 = 0$)? Then the limit becomes:

$$D_u f(0,0) = \lim_{h\to 0} \frac{0}{h^2 u_1^4} = \lim_{h\to 0} 0 = 0$$

And in fact, we have just found that $f_x(0,0) = 0$.

Now, if $u_2 \neq 0$, we can continue where we left off:

$$D_u f(0,0) = \lim_{h\to 0} \frac{u_1^2 u_2}{h u_1^4 + u_2^2} = \frac{u_1^2 u_2}{u_2^2} = \frac{u_1^2}{u_2}$$

So, for example, if $\mathbf{u} = \langle 0, 1 \rangle$, then we get $D_u f(0,0) = f_y(0,0) = 0$. In the direction, say of $\frac{1}{\sqrt{2}}\langle 1, 1 \rangle$, then

$$D_u f(0,0) = \frac{1}{\sqrt{2}}$$

In general, we have shown that the directional derivative exists in all directions.

3. Show that $D_u f(0,0) \neq \nabla f(0,0) \cdot \mathbf{u}$

   SOLUTION: Since $f_x(0,0) = f_y(0,0) = 0$, then the dot product on the right side of the equation is zero for **all** directions $\mathbf{u}$. However, we showed one direction where the directional derivative was not zero.

   CONCLUSION: Since f is not differentiable (or even continuous) at the origin, we cannot use the shortcut formula for computing the directional derivative.

Final comment: These "bad" functions have been set up in a particular way so that they fail. You should rest assured that most functions you will use will be differentiable, and so the shortcut formula **will** work. But, these examples show us the differences between the derivative in one dimension and multiple dimensions.