Solutions to Group Work, 15.3

1. Write double integrals that represent the following areas. For extra practice, write them both as Type I regions and Type II regions.

   (a) The area enclosed by $y = x - x^2$ and $y = x/4$
   SOLUTION: The point of intersection is found by solving $x - x^2 = x/4$, so $x = 3/4$, $y = 3/16$. Then
   As a Type I region (bounded by y as functions of x, or going up/down first):
   $$\int_0^{3/4} \int_{y=x/4}^{y=x-x^2} 1\, dy\, dx$$
   As a Type II region (bounded by x as functions of y, or going left/right first):
   We need to solve $y = x - x^2$. Do this by completing the square or by the quadratic formula (think of y as a constant):
   $$x = \frac{1 \pm \sqrt{1-4y}}{2}$$
   The function with the minus root is the left half of the parabola, and the other half is from the positive root. Therefore, the area is:
   $$\int_{y=0}^{y=3/16} \int_{x=\frac{1-\sqrt{1-4y}}{2}}^{x=4y} 1\, dx\, dy + \int_{y=3/16}^{y=1/4} \int_{x=\frac{1-\sqrt{1-4y}}{2}}^{x=\frac{1+\sqrt{1-4y}}{2}} 1\, dx\, dy$$

   (b) The area enclosed by $y = x^2$ and $y = x^{1/4}$:
   SOLUTION: As a Type I region (then x^2 is the bottom function):
   $$\int_{x=0}^{x=1} \int_{y=x^2}^{y=x^{1/4}} 1\, dy\, dx$$
   As a Type II region (then $y = x^{1/4}$ is the leftmost function):
   $$\int_{y=0}^{y=1} \int_{x=y^4}^{x=\sqrt{y}} 1\, dx\, dy$$

(c) The area enclosed by $y = \sqrt{x}$ and $y = x^4$ and $y = 1/2$.
SOLUTION: Most naturally expressed as a Type II region:

$$\int_{y=0}^{y=1/2} \int_{x=y^2}^{x=y^{1/4}} 1\, dx\, dy$$

However, we could write it as a Type I region. We need the points of intersection between $y = 1/2$ and the two curves:

$$y = \sqrt{x} \quad \Rightarrow \quad (1/4, 1/2)$$

and

$$y = x^4 \quad \Rightarrow \quad x = \sqrt[4]{1/2} \approx 0.84$$

Now, for $0 \le x \le 1/4$, the bottom function is $y = x^4$ and the top function is $y = \sqrt{x}$, and in the interval $1/4 \le x \le \sqrt[4]{1/2}$, the bottom function is still $y = x^4$, but now the top function is $y = 1/2$. Therefore, add them together:

$$\int_0^{1/4} \int_{x^4}^{\sqrt{x}} 1\, dy\, dx + \int_{1/4}^{(1/2)^{1/4}} \int_{x^4}^{1/2} 1\, dy\, dx$$

(d) The area enclosed by $y = x^2$ and $y = (x-2)^2$.
SOLUTION: As a Type I region, the point of intersection is $(1, 1)$:

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} 1\, dy\, dx + \int_{x=1}^{x=2} \int_{y=0}^{y=(x-2)^2} 1\, dy\, dx$$

As a Type II region, note that the right half of the parabola $y = x^2$ can be expressed as $x = \sqrt{y}$, but the LEFT half of $y = (x-2)^2$ should be described as $x = 2 - \sqrt{y}$ (in class, we might have had $x = 2 + \sqrt{y}$, so that would have been an error).

$$\int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=2-\sqrt{y}} 1\, dy\, dx$$

2. Upper hemisphere of a sphere of radius 1.