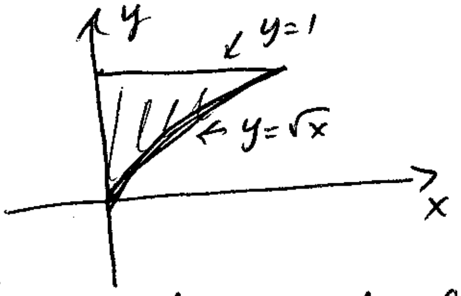
#33 $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$

Re-write as an equivalent integral in the five other orders.

SOLUTION:

It is important to look at the projections in the coordinate planes:

Ⓐ In the xy plane:

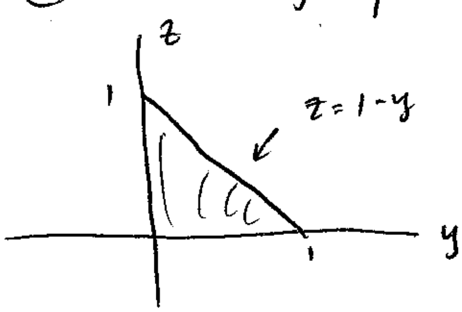


Use for ordering $dz\, dy\, dx$ or $dz\, dx\, dy$.

$dz\, dy\, dx$ is already given; for $dz\, dx\, dy$, we have:

but $\left.\begin{array}{l} 0 \le z \le 1-y \\ 0 \le x \le y^2 \\ 0 \le y \le 1 \end{array}\right\}$ $\int_0^1 \int_0^{y^2} \int_0^{1-y} f\, dz\, dx\, dy.$

Ⓑ The yz plane: Use for $dx\, dy\, dz$ & $dx\, dz\, dy$

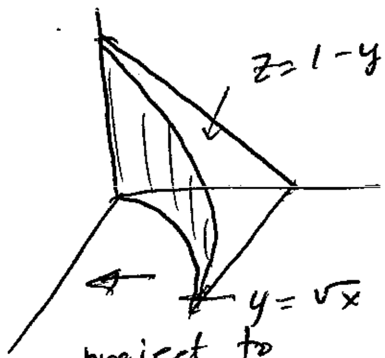


where $0 \le x \le y^2$

$$\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\, dx\, dy\, dz$$

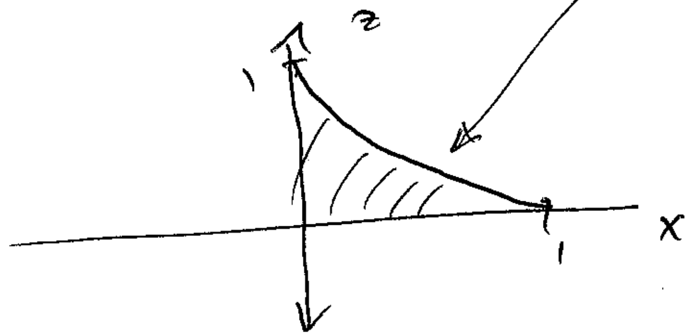
and $\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\, dx\, dz\, dy$

ⓒ In the xz plane: Use for $dy\,dx\,dz$ & $dy\,dz\,dx$



Top curve is found by $z = 1-y$ and $y=\sqrt{x}$ (Remove y):

$z = 1-\sqrt{x}$ or $x = (1-z)^2$



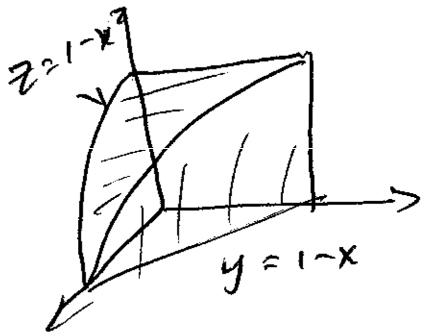
Now, in both $dx\,dz$ & $dz\,dx$, the "heights" y will vary from the cylinder $y=\sqrt{x}$ to the plane $z = 1-y$,

so $\sqrt{x} \le y \le 1-z$

And:

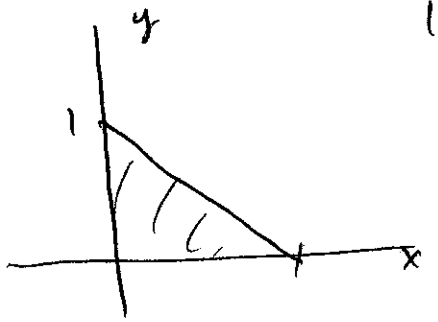
$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

#34 Shown is "$dy\,dz\,dx$", get the other 5 orders of the solid:



As before, consider the 3 projections!

(A) xy plane, for $dz\,dy\,dx$ & $dz\,dx\,dy$:



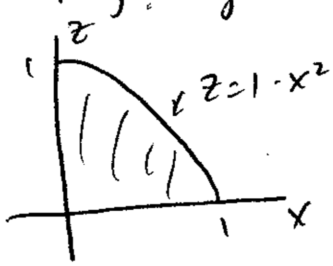
In both cases, over this region, $0 \le z \le 1-x^2$

So:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$

and switching direction: $\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy$

(B) Projecting to the xz plane, (for $dy\,dx\,dz$ & $dy\,dz\,dx$)



$\underline{0 \le y \le 1-x}$

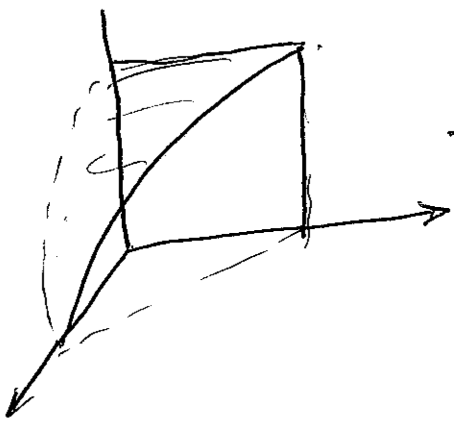
and:

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

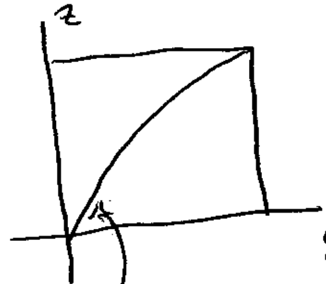
and

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$

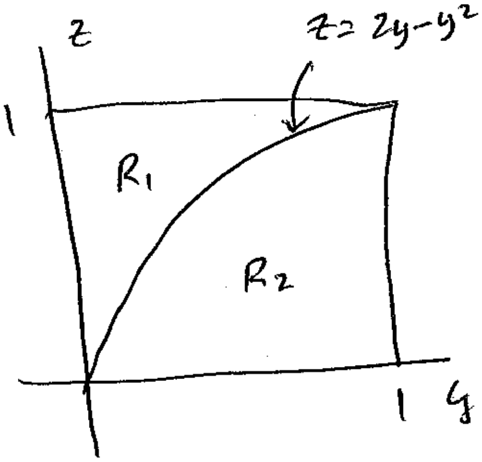
(c) When we project to the yz plane, we have to be a bit more careful –
In particular, where does the edge representing the intersection of $z = 1-x^2$ and $y = 1-x$ go?



Project $\longrightarrow$ to yz plane



This is the intersection of $y = 1-x$ and $z = 1-x^2$ in the yz plane



(Remove x from the intersection)

$x = 1-y \Rightarrow$
$z = 1-(1-y)^2$
$= 2y - y^2$

Break the integral up into a sum:

Over R_1: $0 \le x \le \sqrt{1-z}$:

$$\int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\, dx\, dz\, dy$$

Over R_2: $0 \le x \le 1-y$

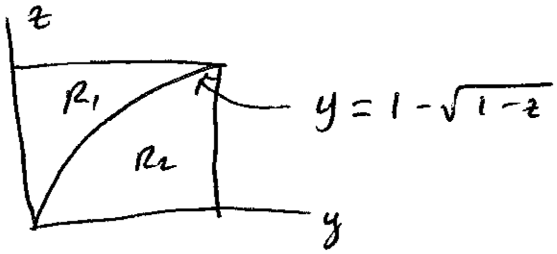
$$\int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x,y,z)\, dx\, dz\, dy$$

(Finish up by writing $dx\, dy\, dz$)

To finish up, note that

$$z = 1-(1-y)^2 \Rightarrow \quad (1-y)^2 = 1-z$$
$$1-y = \pm\sqrt{1-z}$$
$$y = 1 \pm \sqrt{1-z}$$

Our y-values are between 0 & 1, so we choose



and again we write these integrals seperately:

Over R_1: $0 \le x \le \sqrt{1-z}$, so

$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\, dx\, dy\, dz \quad + \quad \text{Integral over } R_2 \text{ below:}$$

$$\int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\, dx\, dy\, dz$$