Exam 2 Sample Solutions

Be sure to look over your old quizzes and homework as well. For limits, we will provide a graph and contours. No calculators will be allowed for this exam.

1. True or False, and explain:

   (a) There exists a function f with continuous second partial derivatives such that

   $$f_x(x,y) = x + y^2 \qquad f_y = x - y^2$$

   SOLUTION: False. If the function has continuous second partial derivatives, then Clairaut's Theorem would apply (and $f_{xy} = f_{yx}$). However, in this case:

   $$f_{xy} = 2y \qquad f_{yx} = -2y$$

   (b) The function f below is continuous at the origin.

   $$f(x,y) = \begin{cases} \frac{2xy}{x^2+2y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   SOLUTION: Check the limit- First, how about $y = x$ versus $y = -x$?

   $$\lim_{(x,x)\to(0,0)} \frac{2x^2}{3x^2} = \frac{2}{3} \qquad \lim_{(x,-x)\to(0,0)} \frac{-2x^2}{3x^2} = \frac{-2}{3}$$

   Yep, that did it- The limit does not exist at the origin, therefore the function is not continuous at the origin (it is continuous at all other points in the domain).

   (c) If $\vec{r}(t)$ is a differentiable vector function, then

   $$\frac{d}{dt}|\vec{r}(t)| = |\vec{r}\,'(t)|$$

   SOLUTION: False.

   $$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{2}(\mathbf{r}(t)\cdot\mathbf{r}(t))^{-1/2}\left(\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t)\right) = \frac{\mathbf{r}'(t)\cdot\mathbf{r}(t)}{|\mathbf{r}(t)|}$$

   Extra: If you're not sure about it, try to verify the formula with $\mathbf{r}(t) = \langle 3t^2, 6t-5\rangle$.

   (d) If $z = 1 - x^2 - y^2$, then the linearization of z at $(1,1)$ is

   $$L(x,y) = -2x(x-1) - 2y(y-1)$$

   SOLUTION: False for two reasons. We have forgotten to evaluate the partial derivatives of f at the base point $(1,1)$ (and so the resulting formula is not linear). We have also forgotten to evaluate the function itself at $(1,1)$. The linearization should be:

   $$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = -1 - 2(x-1) - 2(y-1)$$

   (e) We can always use the formula: $\nabla f(a,b)\cdot\vec{u}$ to compute the directional derivative at (a,b) in the direction of $\vec{u}$.

   SOLUTION: False. This formula only works if f is differentiable at (a,b) (See Exercise 4 below).

   (f) Different parameterizations of the same curve result in identical tangent vectors at a given point on the curve.

   SOLUTION: False. The magnitude of $\mathbf{r}(t)$ is the velocity. For example, $\mathbf{r}(3t)$ will have a magnitude that is three times that of the original- If you want an actual example, consider

   $$\mathbf{r}(t) = \langle\cos(t), \sin(t)\rangle$$

   At the point on the unit circle $(1/\sqrt{2}, 1/\sqrt{2})$, the magnitude of $\mathbf{r}'(\pi/4) = 1$. Replace t by $3t$ (and evaluate at $t = \pi/12$ to have the same point on the curve), and the speed is 3 instead of 1.

   Why did we bring this up? If we re-parameterize with respect to *arc length*, the velocity is always 1 unit (so at $s = 1$, you've traveled one unit of length, etc).

(g) If $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions, then

$$\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}\,'(t)$$

SOLUTION: False. It looks like the product rule:

$$\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$$

(h) If $f_x(a,b)$ and $f_y(a,b)$ both exist, then f is differentiable at (a,b).
SOLUTION: False. Our theorem says that in order to conclude that f is differentiable at (a,b), the partial derivatives must be *continuous* at (a,b). Just having the partial derivatives exist at a point is a weak condition- It is not enough to even have continuity.

(i) At a given point on a curve $(x(t_0), y(t_0), z_0(t))$, the osculating plane through that point is the plane through $(x(t_0), y(t_0), z(t_0))$ with normal vector is $\vec{B}(t_0)$.
SOLUTION: True (by definition).

2. Show that, if $|\vec{r}(t)|$ is a constant, then $\vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$. (HINT: Differentiate $|\vec{r}(t)|^2 = k$)

SOLUTION: Using the hint,

$$0 = \frac{d}{dt}k = \frac{d}{dt}\left(|\vec{r}(t)|^2\right) = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 2\vec{r}\,'(t)\cdot\vec{r}(t)$$

Therefore, the dot product is zero (and so $\vec{r}\,'(t)$ and $\vec{r}(t)$ are orthogonal).

3. Reparameterize the curve with respect to arc length measuring from $t = 0$ in the direction of increasing t:

$$\mathbf{r} = 2t\mathbf{i} + (1 - 3t)\mathbf{j} + (5 + 4t)\mathbf{k}$$

SOLUTION: Find s as a function of t, invert it then substitute it back into the expression so that $\mathbf{r}$ is a function of s. In this case,

$$s = \int_0^t |\mathbf{r}'(u)|\,du = \sqrt{29}\,t$$

Therefore, $t = s/\sqrt{29}$, and

$$\mathbf{r}(s) = \left\langle \frac{2}{\sqrt{29}}s, 1 - \frac{3}{\sqrt{29}}s, 5 + \frac{4}{\sqrt{29}}s \right\rangle$$

4. Is it possible for the directional derivative to exist for every unit vector $\vec{u}$ at some point (a,b), but f is still not differentiable there?

Consider the function $f(x,y) = \sqrt[3]{x^2y}$. Show that the directional derivative exists at the origin (by letting $\vec{u} = \langle\cos(\theta), \sin(\theta)\rangle$ and using the **definition**), BUT, f is not differentiable at the origin (because if it were, we could use $\nabla f \cdot \vec{u}$ to compute $D_{\vec{u}}f$).

SOLUTION: Compute the directional derivative at the origin by using the definition:

$$D_u f(0,0) = \lim_{h\to 0}\frac{f(0 + h\cos(\theta), 0 + h\sin(\theta)) - f(0,0)}{h} = \lim_{h\to 0}\frac{h\sqrt[3]{\cos^2(\theta)\sin(\theta)}}{h} = \sqrt[3]{\cos^2(\theta)\sin(\theta)}$$

Notice that by using the definition, $\theta = 0$ corresponds to the rate of change parallel to the $x-$axis, and $\theta = \pi/2$ is the rate of change parallel to the $y-$axis:

$$f_x(0,0) = 0 \qquad f_y(0,0) = 0$$

so that, if f were differentiable at the origin, we could use

$$D_u f = \nabla f \cdot \vec{u} = 0$$

for every vector $\vec{u}$, but that is not the case (our directional derivative is not always zero).

5. If $f(x, y) = \sin(2x + 3y)$, then find the linearization of f at $(-3, 2)$.

SOLUTION: We have $f(-3, 2) = \sin(0) = 0$ and

$$f_x(x, y) = 2\cos(2x + 3y) \quad \Rightarrow \quad f_x(-3, 2) = 2$$

$$f_y(x, y) = 3\cos(2x + 3y) \quad \Rightarrow \quad f_y(-3, 2) = 3$$

Therefore,

$$L(x, y) = 0 + 2(x + 3) + 3(y - 2) = 2(x + 3) + 3(y - 2)$$

6. The radius of a right circular cone is increasing at a rate of 3.5 inches per second while its height is decreasing at a rate of 4.3 inches per second. At what rate is the volume changing when the radius is 160 inches and the height is 200 inches? ($V = \frac{1}{3}\pi r^2 h$)

SOLUTION:

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

Use $r = 160$, $h = 200$, $dr/dt = 3.5$ and $dh/dt = -4.3$, $dV/dt \approx 37973.3\pi$

7. Find the differential of the function: $v = y\cos(xy)$

SOLUTION:

$$dv = v_x\,dx + v_y\,dy = (-y^2\sin(xy)\,dx + (\cos(xy) - xy\sin(xy))\,dy$$

8. Find the maximum rate of change of $f(x, y) = x^2 y + \sqrt{y}$ at the point $(2, 1)$, and the direction in which it occurs.

SOLUTION: The maximum rate of change occurs if we move in the direction of the gradient. We see this by recalling that:

$$D_u f = \nabla f \cdot \vec{u} = |\nabla f|\cos(\theta)$$

so we find the gradient at the point $(2, 1)$

$$\nabla f = \langle 4, 9/2 \rangle$$

So if we move in that direction, the we get the max rate of change, which is

$$|\nabla f| = \sqrt{4^2 + \frac{81}{4}} = \frac{\sqrt{145}}{2} \approx 6.02$$

9. Find an expression for

$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right]$$

SOLUTION:

$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right] = \mathbf{u}' \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})'$$

Taking this derivative, we see that

$$\frac{d}{dt}\left[\mathbf{u}(t) \cdot (\mathbf{v}(t) \times \mathbf{w}(t)\right] = \mathbf{u}' \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot (\mathbf{v}' \times \mathbf{w} + \mathbf{v} \times \mathbf{w}')$$

10. Use Lagrange Multipliers to find the maximum and minimum of f subject to the given constraints:

$$f(x, y) = x^2 y \qquad x^2 + y^2 = 1$$

SOLUTION:

At optimality, the gradients are parallel, so the system of equations we are solving is given below:

$$\begin{aligned} 2xy &= 2\lambda x \\ x^2 &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned}$$

If $x \neq 0$ in the first equation, then $\lambda = y$. Going to the second equation, that implies that $x^2 = 2y^2$. Now to the third equation, we can solve for y, and therefore also x:

$$3y^2 = 1 \quad \Rightarrow \quad y = \pm\sqrt{\frac{1}{3}} \quad \Rightarrow \quad x = \pm\sqrt{\frac{2}{3}}$$

Are there any other solutions? If $x = 0$ in the first equation, then $y = \pm 1$ from the third equation- in that case, $f(0, \pm 1) = 0$.

Substitute into f and we find the max and min:

$$f\left(\pm\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}\right) = \frac{2}{3\sqrt{3}} \qquad f\left(\pm\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}\right) = \frac{-2}{3\sqrt{3}}$$

11. The curves below intersect at the origin. Find the angle of intersection to the nearest degree:

$$\vec{r}_1(t) = \langle t, t^2, t^9 \rangle \quad \vec{r}_2(t) = \langle \sin(t), \sin(5t), t \rangle$$

SOLUTION:

The angle of intersection is the angle between the tangent vectors at the origin. First differentiate, then evaluate at $t = 0$:

$$\vec{r}_1{}'(t) = \langle 1, 2t, 9t^8 \rangle \quad \vec{r}_2{}'(t) = \langle \cos(t), 5\cos(5t), 1 \rangle \quad \Rightarrow \quad \langle 1, 0, 0 \rangle, \langle 1, 5, 1 \rangle$$

To find the angle, we use the relationship:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos(\theta)$$

In our case:

$$\cos(\theta) = \frac{1}{\sqrt{1^2 + 5^2 + 1^2}} \quad \Rightarrow \quad \theta = \cos^{-1}(1/\sqrt{27}) \approx 79^\circ$$

12. Find three positive numbers whose sum is 100 and whose product is a maximum.

SOLUTION: Let x, y, z be the three numbers. Then we want to find the maximum of $P(x, y, z) = xyz$ subject to the constraint that $x + y + z = 100$ and they are all positive.

Alternative 1: Let $P(x, y) = xy(100 - x - y)$. The critical points are

$$\begin{aligned} y(100 - x - y) - xy &= 0 \\ x(100 - x - y) - xy &= 0 \end{aligned}$$

From the first equation, $y = 100 - 2x$ (we can throw out $y = 0$). Substitute this into the second equation to find that $x = 100/3$. Therefore, $y = 100/3$ and $z = 100/3$. These are the three numbers we wanted.

Alternative 2: Using Lagrange Multipliers,

$$\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \\ x + y + z &= 100 \end{aligned}$$

From the first three equations, if we do not allow zero (then $P = 0$), we have $x = y = z$. Substitute into the fourth equation to see that $x = y = z = 100/3$.

13. Find the equation of the tangent plane and normal line to the given surface at the specified point:

$$x^2 + 2y^2 - 3z^2 = 3 \qquad (2, -1, 1)$$

SOLUTION: This is an implicitly defined surface of the form $F(x, y, z) = k$, therefore, we know that ∇F is orthogonal to the tangent planes on the surface. Compute ∇F at $(2, -1, 1)$, and construct the plane and line:

$$F_x = 2x \quad F_y = 4y \quad F_z = -6z \quad \Rightarrow \quad \nabla F(2, -1, 1) = \langle 4, -4, -6 \rangle$$

Thus, the tangent plane is:

$$4(x-2) - 4(y+1) - 6(z-1) = 0$$

The normal line goes in the direction of the gradient, starting at the given point. In parametric form,

$$x(t) = 2 + 4t \quad y(t) = -1 - 4t \quad z(t) = 1 - 6t$$

14. If $z = x^2 - y^2$, $x = w + 4t$, $y = w^2 - 5t + 4$, $w = r^2 - 5u$, $t = 3r + 5u$, find $\partial z/\partial r$.

    SOLUTION: Use a chart to keep track of the variables; see the solution attached.

15. If $x^2 + y^2 + z^2 = 3xyz$ and we treat z as an implicit function of x, y, then find $\partial z/\partial x$ and $\partial z/\partial y$.

    SOLUTION: Let us define $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$ in keeping with the notation from the text. Then we compute:

$$F(x,y,z) = 0 \Rightarrow F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(2x - 3yz)}{2z - 3xy}$$

    Similarly, we can show that

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(2y - 3xz)}{2z - 3xy}$$

16. If $\mathbf{a}(t) = -10\mathbf{k}$ and $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j}$, find the velocity and position vector functions.

    SOLUTION:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \langle 0, 0, -10t \rangle + \mathbf{v}_0 = \langle 1, 1, -10t - 1 \rangle$$

    And antidifferentiate once more:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \langle t, t, -5t^2 - t \rangle + \mathbf{r}_0 = \langle t + 2, t + 3, -5t^2 - t \rangle$$

17. Find the equation of the normal line through the level curve $4 = \sqrt{5x - 4y}$ at $(4, 1)$ using a gradient.

    SOLUTION: The gradient of g is orthogonal to its level curve at $\sqrt{5x - 4y} = 4$. Find the gradient of g at $(4, 1)$:

$$\nabla g = \frac{1}{8} \langle 5, -4 \rangle$$

    For the line, we simply need to move in the direction of the gradient, so we can simplify the direction to $\langle 5, -4 \rangle$ (not necessary, but easier for the algebra).

    Therefore, the line (in parametric and symmetric form) is:

$$x(t) = 4 + 5t \quad y(t) = 1 - 4t \qquad \text{or} \qquad \frac{x-4}{5} = \frac{y-1}{-4}$$

    Notice that the slope is $-4/5$. If we wanted to check our answer, we could find the slope of the tangent line:

$$5x - 4y = 16 \quad \Rightarrow \quad 5 - 4\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{5}{4}$$

18. Find all points at which the direction of fastest change in the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\vec{i} + \vec{j}$.

    SOLUTION: The direction of the fastest increase is in the direction of the gradient. Therefore, another way to phrase this question is: When is the gradient pointing in the direction of $\langle 1, 1 \rangle$ (very reminiscent of the Lagrange Multiplier):

$$\nabla f = k\langle 1, 1 \rangle \quad \Rightarrow \quad \langle 2x - 2, 2y - 4 \rangle = \langle k, k \rangle$$

    So $k = 2x - 2$ and $k = 2y - 4$, therefore, the points are on the line $2x - 2 = 2y - 4$, or $y = x + 1$. Our conclusion: There are an infinite number of possibilities- All of the form $(a, a + 1)$, which result in the gradient:

$$(2a - 2)\langle 1, 1, \rangle \qquad a > 1$$

19. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

SOLUTION: The volume is $V = xyz$ subject to the constraint that $x + 2y + 3z = 6$.

Alternative 1: Take $V = \frac{1}{3}xy(6 - x - 2y)$, then find the CPs. The only CP with no zeros is $x = 2, y = 1, z = 2/3$.

Alternative 2: Use Lagrange Multipliers to get the same answer (probably much quicker).

20. Find and classify the critical points:

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

SOLUTION: The partial derivatives are:

$$f_x = 3x^2 - 3y \quad f_y = 3y^2 - 3x \quad f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -3$$

The critical points are where $x^2 = y$ and $y^2 = x$, so x, y are both zero or positive:

$$x^4 = x \quad \Rightarrow \quad x^4 - x = 0 \quad \Rightarrow \quad x(x^3 - 1) = 0$$

so $x = 0, y = 0$ or $x = 1, y = 1$. Put these points into the Second Derivatives Test:

$$f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = -9 < 0 \quad \Rightarrow \quad \text{The origin is a SADDLE}$$

$$f_{xx}(1,1)f_{yy}(1,1) - f_{xy}^2(1,1) = 36 - 9 > 0 \quad f_{xx}(1,1) > 0 \Rightarrow \quad \text{Local MIN}$$

21. Let $f(x, y) = x - y^2$. Find the gradient of f at $(3, -1)$. We said that this gradient was perpendicular to a level curve of f- Which one? Draw a sketch showing the level curve and the gradient vector, then find the equation of the tangent line to the level curve and the equation of the normal line.

SOLUTION: The level curve is the one that contains the given point (in this case, $(3, -1)$): Substitute to get $3 - 1 = 2$, so the curve is $x - y^2 = 2$ or $x - 2 = y^2$, which is a sideways parabola shifted to the right two units. The gradient at $(3, -1)$ is $\langle 1, 2\rangle$. The tangent line has slope $-1/2$:

$$\text{Tangent Line: } 1(x + 3) + 2(y + 1) = 0 \text{ or } y + 1 = -\frac{1}{2}(x - 3)$$

$$\text{Normal Line:} x = 3 + t, y = -1 + 2t \text{ or } \frac{y + 1}{2} = (x - 3) \text{ or } y + 1 = 2(x - 3)$$

22. Find the equation of the tangent plane to the surface implicitly defined below at the point $(1, 1, 1)$:

$$x^3 + y^3 + z^3 = 9 - 6xyz$$

SOLUTION: First write this as $F(x, y, z) = 0$, then the gradient of F is the normal vector for the plane (evaluated at $(1, 1, 1)$):

$$F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 9 = 0$$

Then, at $(1, 1, 1)$

$$F_x = 3x^2 + 6yz \quad F_y = 3y^2 + 6xz \quad F_z = 3z^2 + 6xy \quad \Rightarrow \quad \nabla F = \langle 9, 9, 9\rangle$$

Therefore, the tangent plane is:

$$9(x - 1) + 9(y - 1) + 9(z - 1) = 0$$

Alternate and Note: We could have used the following formulas to compute the tangent plane:

$$z_x = \frac{-F_x}{F_z} = -1 \qquad z_y = \frac{-F_y}{F_z} = -1$$

to write the tangent plane as:

$$z - 1 = -(x - 1) - (y - 1)$$

23. Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and $z = 4$. (Hint: In which direction should the tangent line go?)

    The curve is $2x^2 - y^2 = 4$, which is an ellipse (at height 4 in 3-d). The gradient is $\langle 4x, -2y \rangle$ so at $(-2, 2)$, the gradient is $< -8, -4 >$, so the tangent line (in the xy plane) is:

$$-8(x+2) - 4(y-2) = 0 \quad \text{or} \quad y = -2(x+2) + 2$$

    The tangent line can be expressed as:

$$\langle t, -2(t+2) + 2, 4 \rangle$$

    (But there are multiple ways of expressing it).

24. Find and classify the critical points:

$$f(x, y) = x^3 - 3x + y^4 - 2y^2$$

    SOLUTION: We use the second derivatives test to classify the critical points as local min, local max or saddle.

    Solving for the CPs, we get:

$$f_x(x, y) = 3x^2 - 3 = 0 \qquad f_y(x, y) = 4y^3 - 4y = 0$$

    from which we get $x = \pm 1, y = \pm 1$ and $x = \pm 1, y = 0$ Continuing with second derivatives,

$$D(x, y) = \left| \begin{array}{cc} f_x x & f_x y \\ f_y x & f_y y \end{array} \right] = \left| \begin{array}{cc} 6x & 0 \\ 0 & 12y^2 - 4 \end{array} \right] = 24x(3y^2 - 1)$$

    We'll arrange the results in a list:

Point	D and Classification
$(1, 1)$	$48:$ Local Min
$(1, -1)$	$48:$ Local Min
$(1, 0)$	$-24:$ Saddle
$(-1, 1)$	$-48:$ Saddle
$(-1, -1)$	$-48:$ Saddle
$(-1, 0)$	$24:$ Local Max