Review Questions: Calc III

For the final exam, you may bring a 3" $\times$ 5" card of notes (both sides) with you. You should bring a calculator. To study, please be sure to look over the old exams, old quizzes, then you might look at a homework problem or two over the sections that you may be fuzzy on.

1. Write the parametric form for either the given curve or the given surface. In addition, find the domain (if not the natural domain), and the arc length term: ds or the surface area term dS.

   (a) S is the upper half of a sphere of radius k For extra practice, try both Cartesian and Cylindrical. You could do Spherical, but it is computationally extensive.

   (b) C is the curve is the intersection between the plane $x+y+z=1$ and the cylinder $x^2+y^2=9$.

   (c) S is the part of the plane $x+y+z=1$ in the first octant.

   (d) C is the upper semicircle that starts at $(0,1)$ and ends at $(2,1)$.

   (e) S is the part of the cone $z=\sqrt{x^2+y^2}$ beneath the plane $z=1$ with downward orientation.

   (f) S is the cylindrical surface $y=z^2$ for $-1\le z\le 2$, and $0\le x\le 4$.

2. If $\vec{F}$ is a vector field, what is meant by $\mathrm{div}(\vec{F})$ at a point P? (Your answer should include a couple of easy examples). If the vector field is the one given below, find the divergence.
$$\vec{F} = y\mathrm{e}^{x^2}\vec{i} + xy\mathrm{e}^{y}\vec{j} + z\cos(xy)\vec{k}$$

3. If $\vec{F}$ is a vector field, what is meant by $\mathrm{curl}(\vec{F})$ at a point P? To help, consider the vector field $\langle -y, x, 0\rangle$, which is a rotation counterclockwise (if you look straight down at the xy plane). Another vector field is $\langle y, -x, 0\rangle$, which rotates clockwise.

4. An oceanographic vessel suspends a paraboloid shaped net whose shape is roughly $z=\frac{1}{2}(x^2+y^2)$, where the height of the net is 50.

   Water is flowing with velocity
$$\vec{F} = 2xz\vec{i} - (60 + x\mathrm{e}^{-x^2})\vec{j} + z(60-z)\vec{k}$$

   (a) Write down an iterated integral I_1 for the flux of the water through the surface of the net (oriented outward). Include the limits of integration but do not evaluate.

   (b) Use the Divergence Theorem to compare this integral with the flux I_2 across the circular disk which is the open top of the paraboloid-shaped net, and use this to evaluate I_1.

5. Evaluate $\displaystyle\iint_R (x+y)\mathrm{e}^{x^2-y^2}\, dA$, by changing coordinates, if R is the rectangle enclosed by the lines

$$y - x = 0, \qquad y - x = 2, \qquad x + y = 0 \qquad x + y = 3$$

and use the change of coordinates $u = x - y$ and $v = x + y$.

6. Find the limit, if it exists:

$$\lim_{x\to 2} \frac{x^2 - 6x + 8}{x - 2} \qquad \lim_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4 + y^4} \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

What is the most salient difference between the first limit and the other two?

7. Find the projection of the vector $\langle 1, 4, 6\rangle$ onto the vector $\langle -2, 5, -1\rangle$. If we take a unit vector $\vec{x}$ and project it onto $\langle -2, 5, -1\rangle$, for what $\vec{x}$ would the projection have the smallest magnitude? The largest magnitude?

8. Find the local maximum and minimum values and saddle point(s) of the function: $f(x, y) = x^3y + 12x^2 - 8y$.

9. Same function as in 3, but find the global maximum if $-1 \le x \le 1$ and $-1 \le y \le 1$.

10. Suppose E is the region inside the cylinder $x^2+y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

    (a) Find the volume using an appropriate triple integral (Yes, it is easy to find geometrically, so verify your answer!).

    (b) Find parameterization(s) of the surface and write the integral(s) for the surface area (Yes, it is easy to find geometrically- Verify your answer!)

11. Find the area of the parallelogram formed by the vectors $\langle 6, 3, -1\rangle$, $\langle 0, 1, 2\rangle$. Find the volume of the parallelepiped if we add a third vector, $\langle 4, -2, 5\rangle$

12. Is a function differentiable if the partial derivatives both exist at a point? Before you answer, consider the following example:

    Let $f(x) = x^{1/3}y^{1/3}$

    (a) If $x \neq 0$, compute $f_x(x, y)$ (similarly for $f_y(x, y)$).

    (b) Use the **definition** of $f_x(0, 0)$ to show that the partial derivative at $(0, 0)$ is zero (similarly, show it for $f_y(0, 0)$.

    The graph of f would show you that it is not locally linear at the origin.

13. If the partial derivatives for a function exist at a point, does that mean that the function is continuous there? Before you answer, consider the following example:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   (a) Show that f is not continuous at the origin.

   (b) Show, using the definition, that $f_x(0,0) = 0$, and $f_y(0,0) = 0$

   (NOTE: Since f is not continuous at $(0,0)$, it is also not differentiable at $(0,0)$ even though the partial derivatives exist there).

14. We used the theorem in place of the definition for differentiability (Theorem 8, Sect 14.4): Using it, show that $f(x) = x^{1/3}y^{1/3}$ is not differentiable at the origin.

15. True or False?

   (a) If f is differentiable at (a,b) then f is continuous at (a,b).

   (b) If f is not continuous at (a,b), then f cannot be differentiable at (a,b).

   (c) If f is not continuous at (a,b), then f_x and/or f_y cannot exist at (a,b).

16. If $z = x^2 - xy + 3y^2$, and (x,y) changes from $(3,-1)$ to $(2.96,-0.95)$, compare the values of Δz and dz.

17. Find the equation of the tangent plane to $z = \frac{2x+3}{4y+1}$ at $(0,0)$. Would this be the same thing as linearization?

18. If $u = \sqrt{r^2+s^2}$, $r = y + x\cos(t)$ and $s = x + y\sin(t)$, compute $\partial u/\partial x$, $\partial u/\partial y$ and $\partial u/\partial t$ when $x = 1$, $y = 2$ and $t = 0$.

19. Show that the direction in which the rate of change of f is greatest is in the direction of the gradient. You should start with:

$$D_{\vec{u}}f(x,y,z) = \nabla f \cdot \vec{u}$$

   What *is* the greatest rate of change of f if you go in that direction?

   Illustrate your answer with the following example: $f(x,y,z) = 5x^2 - 3xy + xyz$ at the point $P(3,4,5)$.

20. Let $yz = \ln(x+z)$. Find the equations of the tangent plane and normal line to the surface at $(0,0,1)$.

21. If $g(x,y) = x^2 + y^2 - 4x$, find the gradient $\nabla g(1,2)$ and use it to find the tangent line to the level curve $g(x,y) = 1$ at the point $(1,2)$. Sketch the level curve, the tangent line and the gradient vector.

22. Let the curve C be defined parametrically by: $x = t^2$ and $y = t^4 - 1$. Find the equation of the tangent line at $(4, 15)$.

23. Find the work:

    (a) of the vector field $\vec{F} = \langle x, -z, y \rangle$ acting on a particle along the path $\vec{r}(t) = \langle 2t, 3t, -t^2 \rangle$, for $-1 \leq t \leq 1$.

    (b) of the constant force $\vec{F} = \langle 8, -6, 9 \rangle$ that moves an object from the point $(0, 10, 8)$ to $(6, 12, 20)$ along a straight line.

    (c) of the vector field $\vec{F} = \langle 3y - \mathrm{e}^{\sin(x)}, 7x + \sqrt{y^4 + 1} \rangle$ on a particle going around the curve C, which in this case is a circle of radius 3 (assume CCW).

    (d) of the vector field $\vec{F} = \langle -y^2, x, z^2 \rangle$, and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ (C is CCW from above).

24. A region E is a tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 2)$, $(0, 1, 0)$ and $(1, 1/2, 0)$.

    (a) Find the three planes representing the three faces of E.

    (b) Find six integrals that would give the volume of E. (NOTE: Careful in looking at the projection into the yz plane- there are actually two regions to consider).

25. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 y$ subject to the constraint $x^2 + 2y^2 = 6$.

26. Set up an integral to determine the arc length of one period of the sine function (do not evaluate).

27. Use Stokes' Theorem to find the flux of the curl of $\mathbf{F}$ through the surface S, if

    $$\mathbf{F} = \langle xz, yz, xy \rangle$$

    and surface S is the part of the sphere $x^2 + y^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 1$ and above the xy plane.

28. Use Green's Theorem to evaluate $\int_C x^2 y \, dx - xy^2 \, dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

29. Find the flux across the surface:

    $$\vec{F} = \langle xy, yz, zx \rangle$$

    where the surface is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

30. Look over graphical problems: p. 1107, 1; p. 1104, 19; p. 1068, 9-11; p. 1053, 11; p. 1044, 17-18; p. 999, 33; p. 940, 1; p. 930, 3-4; p. 890, 70 (a-c); p. 889, 5-7.