Review SOLUTIONS

For the final exam, you may bring a 3"× 5" card of notes (both sides) with you. You should bring a calculator. To study, please be sure to look over the old exams, old quizzes, then you might look at a homework problem or two over the sections that you may be fuzzy on.

1. Write the parametric form for either the given curve or the given surface. In addition, find the domain (if not the natural domain), and the arc length term: ds or the surface area term dS.

   (a) S is the upper half of a sphere of radius k For extra practice, try both Cartesian and Cylindrical. You could do Spherical, but it is computationally extensive.

   SOLUTION:

   - Cartesian: The equation of the sphere is $x^2 + y^2 + z^2 = k^2$.
     S is parameterized by $\langle x, y, \sqrt{k^2 - x^2 - y^2} \rangle$
     The domain is the set of (x, y) such that $x^2 + y^2 \leq k^2$
     The surface area term is the magnitude of:

   $$\vec{r}_x \times \vec{r}_y = \left\langle \frac{x}{\sqrt{k^2 - x^2 - y^2}}, \frac{y}{\sqrt{k^2 - x^2 - y^2}}, 1 \right\rangle \quad \Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \frac{k}{\sqrt{k^2 - x^2 - y^2}}$$

   (We assume $k > 0$).

   - Cylindrical: S is parameterized by $\langle r\cos(\theta), r\sin(\theta), \sqrt{k - r^2} \rangle$
     The domain is the set of (r, θ) such that $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$
     The surface area term is computed by first computing the derivatives:

   $$\vec{r}_r = \left\langle \cos(\theta), \sin(\theta), \frac{-r}{\sqrt{k - r^2}} \right\rangle \qquad \vec{r}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$$

   The cross product is a bit messy, but the magnitude is nice:

   $$|\vec{r}_r \times \vec{r}_\theta| = r\frac{k}{\sqrt{k^2 - r^2}}$$

   (Note that the extra r you would get from converting the integral from Cartesian to Cylindrical appears automatically!)

   - Spherical (not recommended in this problem- Computationally extensive): S is parameterized by $\langle k\sin(\phi)\cos(\theta), k\sin(\phi)\sin(\theta), k\cos(\phi) \rangle$
     The domain is the set of (ϕ, θ) so that $0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$.
     The surface area term is the magnitude of the normal vector (LOTS of algebra):
     $$|\vec{r}_\phi \times \vec{r}_\theta| = k^2 \sin(\phi)$$

   (Notice again the appearance of the integrand you would use if converting to spherical).

In every case, the surface area integral would come out to $2\pi k^2$

(b) C is the curve is the intersection between the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$.

SOLUTION: Looking straight down the $z-$axis to the xy plane, we see that the x and y coordinates can be parameterized by the circle: $x = 3\cos(t)$ and $y = 3\sin(t)$. Taking into account the heights, $z = 1 - x - y$, substitute in the parameterization of x, y:

$$x(t) = 3\cos(t) \qquad y(t) = 3\sin(t) \qquad z(t) = 1 - 3(\cos(t) + \sin(t))$$

The arc length term is a little messy; on the exam I probably would tell you not to simplify:

$$ds = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2 + (1 - 3(cos(t) + \sin(t)))^2}\, dt$$

(c) S is the part of the plane $x + y + z = 1$ in the first octant.
SOLUTION: $\vec{r}(x, y) = \langle x, y, 1 - x - y \rangle$. (Draw a sketch to see the domain) The domain is the set of (x, y) bounded by $x = 0, y = 0$ and $x + y = 1$.
The surface area term is:

$$\vec{r}_x \times \vec{r}_y = \langle 1, 1, 1 \rangle \quad \Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{3}$$

(d) C is the upper semicircle that starts at $(0, 1)$ and ends at $(2, 1)$.
SOLUTION: (The wording did not make this clear- Sorry! Make it the right half of the circle with center at $(1, 1)$ and radius 1.
With that clarification, we see that one way of parameterizing the curve is:

$$\vec{r}(t) = \langle \cos(t) + 1, \sin(t) + 1 \rangle \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

(e) S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation.

SOLUTION: We're all set as $z = f(x, y)$. Note the domain is $x^2 + y^2 \le 1$. And the surface term is the usual (do use the shortcuts where you can). You might notice that this is the negative of the usual formal- That's because we want a downward pointing normal.

$$\vec{r}_x \times \vec{r}_y = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right\rangle$$

Multiplying by -1 doesn't change the magnitude though: $\sqrt{2}$.

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(f) S is the cylindrical surface $y = z^2$ for $-1 \le z \le 2$, and $0 \le x \le 4$.

SOLUTION: Use x, z as the independent coordinates (and the domain is given to you!).

$$\vec{r}(x, z) = \langle x, z^2, z \rangle$$

We weren't given which orientation to choose, so we take the one that comes with the parameterization. You should get that:

$$\vec{r}_x \times \vec{r}_z = \langle 0, -1, 2z \rangle$$

so that the magnitude (for the surface area term) is $\sqrt{1 + 4z^2}\, dA$.

2. If $\vec{F}$ is a vector field, what is meant by $\operatorname{div}(\vec{F})$ at a point P? (Your answer should include a couple of easy examples). If the vector field is the one given below, find the divergence.

$$\vec{F} = ye^{x^2}\vec{i} + xye^y\vec{j} + z\cos(xy)\vec{k}$$

SOLUTION: The divergence of a vector field at a point P is visualized as the spread of fluid through a small box centered at P. If the velocity is larger going in than going out, the divergence is negative. If the velocity vectors for the fluid are smaller going in that out, the divergence is positive. If the divergence is zero, the fluid is said to be incompressible (note: Water is incompressible).

In the example, the divergence is

$$P_x + Q_y + R_z = 2xye^{x^2} + (xe^y + xye^y) + \cos(xy)$$

3. If $\vec{F}$ is a vector field, what is meant by $\operatorname{curl}(\vec{F})$ at a point P? To help, consider the vector field $\langle -y, x, 0 \rangle$, which is a rotation counterclockwise (if you look straight down at the xy plane). Another vector field is $\langle y, -x, 0 \rangle$, which rotates clockwise.

SOLUTION: The curl at a point P is a vector. The vector is the normal vector to the plane of rotation of the vector field at that point. In fact, the magnitude of the vector is twice the radial velocity of the fluid, but you don't need to know that. In the examples, we get vectors $\langle 0, 0, 2 \rangle$ and $\langle 0, 0, -2 \rangle$ (notice that one is positive due to the right hand rule, and one is negative).

4. An oceanographic vessel suspends a paraboloid shaped net whose shape is roughly $z = \frac{1}{2}(x^2 + y^2)$, where the height of the net is 50.

Water is flowing with velocity

$$\vec{F} = 2xz\vec{i} - (60 + xe^{-x^2})\vec{j} + z(60 - z)\vec{k}$$

(a) Write down an iterated integral I_1 for the flux of the water through the surface of the net (oriented outward). Include the limits of integration but do not evaluate.

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SOLUTION: we write $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$ so that the surface for I_1 is in our standard form. Therefore, noting that the water is flowing "down" through the net,

$$\vec{r}_x \times \vec{r}_y = \langle x, y, -1 \rangle$$

Take the dot of this with the vector field $\vec{F}$ and (it doesn't simplify much, so you can leave it as is):

$$I_1 = \iint_{x^2+y^2\leq 100} x^2(x^2+y^2) - y(60 + xe^{-x^2}) - \frac{1}{2}(x^2+y^2)\left(60 - \frac{1}{2}(x^2+y^2)\right)\,dA$$

(b) Use the Divergence Theorem to compare this integral with the flux I_2 across the circular disk which is the open top of the paraboloid-shaped net, and use this to evaluate I_1.

SOLUTION: With I_2 as defined, we see that

$$I_1 + I_2 = \iiint_E \operatorname{div}(\vec{F})\,dV$$

Where in this case, E is the solid paraboloid. We compute the divergence as 60. To integrate over the solid, it is best to use cylindrical coordinates. The domain in the xy plane is the disk $x^2 + y^2 = 100$:

$$\iiint_E \operatorname{div}(\vec{F})\,dV = 60 \int_0^{2\pi} \int_0^{10} \int_{r^2/2}^{50} r\,dz\,dr\,d\theta = 60 \cdot 2500\pi = 150,000\pi$$

To find I_1, we first compute the flux through the top of the paraboloid, labeled I_2. In this case, the normal vector is $\langle 0,0,1 \rangle$, so the dot product with $\vec{F}$ is $60z - z^2$. But, for this surface, z is fixed at 50, so the flux is simply:

$$\iint_{x^2+y^2\leq 100} \vec{F}\cdot(\vec{r}_x \times \vec{r}_y)\,dA = \iint_{x^2+y^2\leq 100} 500\,dA = 50,000\pi$$

And we see that I_1 is $100,000\pi$.

5. Evaluate $\displaystyle\iint_R (x+y)e^{x^2-y^2}\,dA$, by changing coordinates, if R is the rectangle enclosed by the lines

$$y - x = 0, \qquad y - x = 2, \qquad x + y = 0 \qquad x + y = 3$$

and use the change of coordinates $u = x - y$ and $v = x + y$.

NOTE: The first two lines should have been $x - y = 0$ and $x - y = 2$ to match u, but we can do it as is.

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SOLUTION: From the way the problem is set up, we see that

$$-2 \le u \le 0 \qquad 0 \le v \le 3$$

For the substitutions, note that $x + y = u$ and $x^2 - y^2 = uv$. For the Jacobian, we have to solve the equations for x, y in terms of u, v:

$$x = \frac{1}{2}(u + v) \qquad y = \frac{1}{2}(v - u)$$

Therefore the Jacobian is:

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{array}{cc} 1/2 & 1/2 \\ -1/2 & 1/2 \end{array} \right| = \frac{1}{2}$$

We now have it. We might integrate with u first, then we don't need to do integration by parts-

$$\int_0^3 \int_{-2}^0 v e^{uv} \frac{1}{2} \, du \, dv = \frac{1}{2} \int_0^3 1 - e^{-2v} \, dv = \frac{1}{2}(5 + e^{-6})$$

6. Find the limit, if it exists:

$$\lim_{x \to 2} \frac{x^2 - 6x + 8}{x - 2} \qquad \lim_{(x,y) \to (0,0)} \frac{6x^3 y}{2x^4 + y^4} \qquad \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$$

What is the most salient difference between the first limit and the other two?

SOLUTION:

$$\lim_{x \to 2} \frac{x^2 - 6x + 8}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x - 4)}{x - 2} = -2$$

The next limit does not exist, which we can show by trying a couple of different directions: Say $(x, 0)$ versus (x, x):

$$\lim_{(x,y) \to (0,0)} \frac{6x^3 y}{2x^4 + y^4} = \lim_{(x,0) \to (0,0)} \frac{0}{2x^4} = 0$$

But

$$\lim_{(x,y) \to (0,0)} \frac{6x^3 y}{2x^4 + y^4} = \lim_{(x,x) \to (0,0)} \frac{6x^4}{3x^4} = 2$$

The limit does exist in the last case. Multiply by the conjugate:

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1}$$

$$\lim_{(x,y) \to (0,0)} \frac{(x^2 + y^2)(\sqrt{x^2 + y^2 + 1} + 1)}{x^2 + y^2 + 1 - 1} = 2$$

What is the most salient difference between the first limit and the other two?

The limit in multiple dimensions can be much more difficult to compute since the approach can be from ANY direction.

7. Find the projection of the vector $\langle 1, 4, 6 \rangle$ onto the vector $\langle -2, 5, -1 \rangle$. If we take a unit vector $\vec{x}$ and project it onto $\langle -2, 5, -1 \rangle$, for what $\vec{x}$ would the projection have the smallest magnitude? The largest magnitude?

SOLUTION:

$$\text{Proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} = \frac{2}{5} \langle -2, 5, -1 \rangle$$

For the second part, if we take the magnitude (squared) of the projection, we have:

$$|\text{Proj}_{\vec{a}}(\vec{b})|^2 = \left| \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \right| |\vec{a}|^2 = |(\vec{b}) \cdot \vec{a}|$$

In our equation, think of $\vec{b}$ as $\vec{x}$ and $\vec{a}$ as fixed, $\langle -2, 5, -1 \rangle$. Then, we recall that (absolute value included because we're also continuing the equation):

$$|\vec{x} \cdot \vec{a}| = |\vec{x}||\vec{a}| \cos(\theta)|$$

since $|\vec{x}| = 1$ and $|\vec{a}| = \sqrt{30}$, the dot product is a minimum if $\vec{x}$ and $\vec{a}$ are perpendicular (so $\cos(\theta) = 0$) and the (absolute value of the) dot product is a maximum if $\vec{x}$ is parallel to $\rangle -2, 5, -1 \rangle$.

8. Find the local maximum and minimum values and saddle point(s) of the function: $f(x, y) = x^3 y + 12x^2 - 8y$.

SOLUTION: Be sure you recall the Second Derivatives test. In this case,

$$f_x = 3x^2 y + 24x = 3x(xy + 8) \qquad f_y = x^3 - 8$$

The critical points are $x = 2$, $y = -4$. Compute the second derivatives:

$$\begin{vmatrix} f_{xx} = 6xy + 24 & f_{yx} = 3x^2 \\ f_{xy} = 3x^2 & f_{yy} = 0 \end{vmatrix}$$

At the critical point, this determinant is $0 - (f_{xy})^2 = -144$, so the critical point is a saddle point.

9. Same function as in 3, but find the global maximum if $-1 \le x \le 1$ and $-1 \le y \le 1$.

TYPO: Change that to "Same function as the one in the previous problem". Then:

SOLUTION: We don't have a critical point in our region, so we just have to worry about the boundary. We have 4 computations (one for each edge):

$$x = -1, -1 \le y \le 1 \qquad f(-1, y) = -9y + 12$$

The maximum here is 21 at $y = -1$, and the minimum is 3 at $y = 1$. Next,

$$x = 1, -1 \le y \le 1 \qquad f(1, y) = -7y + 12$$

The maximum is 19 at $y = -1$ and the minimum is 5 at $y = 1$. If the other two sides are linear in x, we could stop (the extrema will be at the endpoints). No luck- We have to keep going...

$$y = -1, -1 \le x \le 1 \qquad f(x, -1) = -x^3 + 12x^2 + 8$$

and the derivative is $-3x(x - 8)$. Therefore, the local extrema is at $x = -1, x = 1$, or $x = 0$ (do not include $x = 8$, since it is outside of our domain). At $x = 0$, we get 8. (Notice the values at the endpoints have already been computed).

Finally, with $y = 1$, similar reasoning holds:

$$f(x, 1) = x^3 + 12x^2 - 8$$

The only critical point is $x = 0$, at which we have -8.

Overall solution: The global maximum is 21, occurring at $(-1, -1)$, and the global minimum is -8, occurring at $(0, 1)$.

10. Suppose E is the region inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

   (a) Find the volume using an appropriate triple integral (Yes, it is easy to find geometrically, so verify your answer!).

   SOLUTION: The volume of the cylinder is (geometrically) the area of the face times the hieght (as a cylinder). In this case, the area of a face is the area of a circle of radius 4, and the height is 9: Volume is $16 \times 9\pi = 144\pi$.

   Using Calculus and cylindrical coordinates:

   $$\int_0^{2\pi} \int_0^4 \int_{-5}^4 r \, dz \, dr \, d\theta = 2\pi \cdot \frac{1}{2} 4^2 \cdot (4 - -5) = 144\pi$$

   (b) Find parameterization(s) of the surface and write the integral(s) for the surface area (Yes, it is easy to find geometrically- Verify your answer!)

   SOLUTION: Geometrically, the surface area is twice the area of a circle, plus the area of the cylinder wall. The surface area of the cylinder wall is the height times the circumference, $2\pi r$. From this, we get 72π (then add the areas of the circles to get 104π.

   Using Calculus, we see if we get the same thing:

   The sides of the cylinder can be written parametrically as:

   $$\vec{r}(\theta, z) = \langle 4\cos(\theta), 4\sin(\theta), z \rangle$$

   We compute the surface integral for the cylinder walls:

   $$\vec{r}_\theta = \langle -4\sin(\theta), 4\cos(\theta), 0 \rangle \qquad \vec{r}_z = \langle 0, 0, 1 \rangle$$

so the cross product is: $\vec{r}_\theta \times \vec{r}_z = \langle 4\cos(\theta), 4\sin(\theta), 0 \rangle$, so we integrate the magnitude:

$$\iint\limits_S dS = \int_0^{2\pi} \int_{-5}^4 \sqrt{4^2 \cos^2(\theta) + 4^2 \sin^2(\theta) + 0}\, dz\, d\theta = 4 \cdot 2\pi \cdot 9 = 72\pi$$

To get the full surface area, add in the area of the top and bottom circles to get 104π

11. Find the area of the parallelogram formed by the vectors $\langle 6, 3, -1 \rangle$, $\langle 0, 1, 2 \rangle$. Find the volume of the parallelepiped if we add a third vector, $\langle 4, -2, 5 \rangle$

(Vector labels added later)

SOLUTION: The area is the magnitude of the cross product.

$$|\vec{a} \times \vec{b}| = |\langle 7, -12, 6 \rangle| = \sqrt{49 + 144 + 26} = \sqrt{229}$$

The volume of the parallelepiped is the "scalar triple product", or since we have computed the cross product already:

$$|\langle 4, -2, 5 \rangle \cdot \langle 7, -12, 6 \rangle| = 82$$

12. Is a function differentiable if the partial derivatives both exist at a point? Before you answer, consider the following example:

Let $f(x) = x^{1/3} y^{1/3}$

(a) If $x \neq 0$, compute $f_x(x, y)$ (similarly for $f_y(x, y)$).

$$f_x(x, y) = \frac{y^{1/3}}{3x^{2/3}} \qquad f_y(x, y) = \frac{x^{1/3}}{3y^{2/3}}$$

(b) Use the **definition** of $f_x(0, 0)$ to show that the partial derivative at $(0, 0)$ is zero (similarly, show it for $f_y(0, 0)$).

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} 0 = 0$$

$$f_y(0, 0) = \lim_{h \to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \to 0} 0 = 0$$

The graph of f would show you that it is not locally linear at the origin.

We note that the derivatives (in x and y both exist at the origin, but f_x, f_y are not continuous at the origin, and therefore, this function is not differentiable at the origin.

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13. If the partial derivatives for a function exist at a point, does that mean that the function is continuous there? Before you answer, consider the following example:

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Show that f is not continuous at the origin.

We notice that the limit along $y = x$ is $1/2$, but the limit along $x = 0$ is 0. Thus, the limit does not exist at the origin.

(b) Show, using the definition, that $f_x(0, 0) = 0$, and $f_y(0, 0) = 0$

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

Same for f_y.

(NOTE: Since f is not continuous at $(0, 0)$, it is also not differentiable at $(0, 0)$ even though the partial derivatives exist there).

14. We used the theorem in place of the definition for differentiability (Theorem 8, Sect 14.4): Using it, show that $f(x) = x^{1/3}y^{1/3}$ is not differentiable at the origin.

Compute the partial derivatives:

$$f_x(x, y) = \begin{cases} y^{1/3}/(3x^{2/3}) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

If we come in along $y = x$, the limit does not exist (goes to infinity as we come in from the positive side). A similar thing occurs with f_y.

15. True or False?

(a) If f is differentiable at (a, b) then f is continuous at (a, b). **True**, now that "differentiable" means that the partial derivatives are continuous at (a, b).

(b) If f is not continuous at (a, b), then f cannot be differentiable at (a, b). **True** (this is actually the contrapositive of the previous statement, which is logically equivalent).

(c) If f is not continuous at (a, b), then f_x and/or f_y cannot exist at (a, b). **False** See the example in the previous questions.

16. If $z = x^2 - xy + 3y^2$, and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

Recall that dz is used to approximate Δz. If $z = f(x, y)$, then

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

and
$$dz = f_x(x, y)dx + f_y(x, y)dy$$
(where $dx = \Delta x$ and $dy = \Delta y$). In this case, the actual change in z:
$$f(2.96, -0.95) - f(3, -1) = -0.7007$$
With $f_x = 2x - y$ so that $f_x(3, -1) = 7$ and $f_y = -x + 6y$, so that $f_y(3, -1) = -9$, we have:
$$\Delta z = 7 \cdot -0.04 + -9 \cdot 0.05 = -0.73$$

17. Find the equation of the tangent plane to $z = \frac{2x+3}{4y+1}$ at $(0, 0)$. Would this be the same thing as linearization? This is kind of the same thing as linearization, although we sometimes write them differently. To find the tangent plane, we need the partial derivatives:
$$f(0, 0) = 3 \qquad f_x(0, 0) = 2 \qquad f_y(0, 0) = -12$$
The tangent plane:
$$2(x - 0) - 12(y - 0) + (z - 3) = 0 \qquad z = 3 + 2x - 12y$$
The linearization is a function of x, y:
$$L(x, y) = 3 - 2x + 12y$$

18. If $u = \sqrt{r^2 + s^2}$, $r = y + x\cos(t)$ and $s = x + y\sin(t)$, compute $\partial u/\partial x$, $\partial u/\partial y$ and $\partial u/\partial t$ when $x = 1$, $y = 2$ and $t = 0$.

    SOLUTION: Use a tree diagram to help keep track of the variables:

$$
\begin{array}{ccc}
 & u & \\
r & & s \\
x \quad y \quad t & & x \quad y \quad t
\end{array}
\qquad \Rightarrow \qquad
\begin{aligned}
u_x &= u_r r_x + u_s s_x \\
u_y &= u_r r_y + u_s s_y \\
u_t &= u_r r_t + u_s s_t
\end{aligned}
$$

Compute all these numerically with $t = 0, x = 1, y = 2$, then compute $r = 3$ and $s = 1$:
$$u_r = \frac{r}{\sqrt{r^2 + s^2}} = \frac{3}{\sqrt{10}} \qquad u_s = \frac{s}{\sqrt{r^2 + s^2}} = \frac{1}{\sqrt{10}}$$
and differentiate, then evaluate:
$$
\begin{aligned}
r_x &= \cos(t) = 1 & s_x &= 1 \\
r_y &= 1 & s_y &= \sin(t) = 0 \\
r_t &= -x\sin(t) = 0 & s_t &= y\cos(t) = 2
\end{aligned}
$$
Substitute these into our partial derivatives to get:
$$u_x = \frac{3}{\sqrt{10}} \cdot 1 + \frac{1}{\sqrt{10}} \cdot 1 = \frac{4}{\sqrt{10}}$$
Similarly, $u_y = \frac{3}{\sqrt{10}}$ and $u_t = \frac{2}{\sqrt{10}}$

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19. Show that the direction in which the rate of change of f is greatest is in the direction of the gradient. You should start with:

$$D_{\vec{u}}f(x, y, z) = \nabla f \cdot \vec{u}$$

What *is* the greatest rate of change of f if you go in that direction?

SOLUTION:

$$D_{\vec{u}}f(x, y, z) = \nabla f \cdot \vec{u} = |\nabla f| \cdot 1 \cdot \cos(\theta)$$

Therefore, the greatest rate of change is when $\theta = 0$ (and the least is when $\theta = \pi$). The actual rate of change in this case is the magnitude of the gradient.

Illustrate your answer with the following example: $f(x, y, z) = 5x^2 - 3xy + xyz$ at the point $P(3, 4, 5)$.

$$\nabla f = \langle 10x - 3y + yz, -3x + xz, xy \rangle \quad \Rightarrow \quad \nabla f(3, 4, 5) = \langle 38, 6, 12 \rangle$$

If we move in the direction of the gradient, then the instantaneous rate of change is $\sqrt{38^2 + 6^2 + 12^2} = 2\sqrt{406}$

20. Let $yz = \ln(x + z)$. Find the equations of the tangent plane and normal line to the surface at $(0, 0, 1)$.

Think of a "level surface":

$$F(x, y, z) = 0 \quad \text{where} \quad F(x, y, z) = yz - \ln(x + z)$$

and the tangent plane at (a, b, c) is given by (∇F is orthogonal to the level surface):

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

which in our case is:

$$-1(x - 0) + 1(y - 0) - 1(z - 1) = 0 \quad \Rightarrow \quad z = 1 - x + y$$

21. If $g(x, y) = x^2 + y^2 - 4x$, find the gradient $\nabla g(1, 2)$ and use it to find the tangent line to the level curve $g(x, y) = 1$ at the point $(1, 2)$. Sketch the level curve, the tangent line and the gradient vector.

SOLUTION: The gradient is orthogonal to the level curve:

$$\nabla g(x, y) = \langle 2x - 4, 2y \rangle \quad \Rightarrow \quad \nabla g(1, 2) = \langle -2, 4 \rangle$$

We could compute the slope of the tangent line directly, or by taking a direction orthogonal to the gradient. To double check our work, we compute the slope directly (but implicitly):

$$2x + 2y\frac{dy}{dx} - 4 = 0 \quad \frac{dy}{dx} = -\frac{2x - 4}{2y} = \frac{2}{4} = \frac{1}{2}$$

The tangent line: $y - 2 = \frac{1}{2}(x - 1)$. The normal line will be in the direction of the gradient, and so will have slope $4/(-2) = -2$ (which is the negative reciprocal of the slope of the tangent line). The slope of the normal line is $y - 2 = -2(x - 1)$.

Plotting the lines is no problem. To plot the level curve, notice that:

$$x^2 - 4x + y^2 = 1 \quad \Rightarrow \quad x^2 - 4x + 4 + y^2 = 5 \quad \Rightarrow \quad (x - 2)^2 + y^2 = 5$$

so this is a circle of radius $\sqrt{5}$ centered at $(2, 0)$.

22. Let the curve C be defined parametrically by: $x = t^2$ and $y = t^4 - 1$. Find the equation of the tangent line at $(4, 15)$.

SOLUTION: The slope of the tangent line is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{2t} = 2t^2$$

To be at $(4, 15)$, the time value must be $t = 2$, so our slope is 8. Therefore, the tangent line has equation:

$$y - 15 = 8(x - 4)$$

(To verify, this setup was easy enough that we could put it back as $y = f(x)$. In this case, $y = x^2 - 1$.)

23. Find the work:

(a) of the vector field $\vec{F} = \langle x, -z, y \rangle$ acting on a particle along the path $\vec{r}(t) = \langle 2t, 3t, -t^2 \rangle$, for $-1 \le t \le 1$.

SOLUTION: The curl is $2\vec{i}$, so the vector field is not conservative. Also, the curve is not closed, so we cannot use Stokes' Theorem. Therefore, we should go ahead and compute the line integral directly:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} 2x - 3z - 2ty \, dt = \int_{-1}^{1} 4t - 3t^2 \, dt = -2$$

(b) of the constant force $\vec{F} = \langle 8, -6, 9 \rangle$ that moves an object from the point $(0, 10, 8)$ to $(6, 12, 20)$ along a straight line.

SOLUTION: We take $\vec{F} \cdot \langle 6, 2, 12 \rangle = 144$

(c) of the vector field $\vec{F} = \langle 3y - e^{\sin(x)}, 7x + \sqrt{y^4 + 1} \rangle$ on a particle going around the curve C, which in this case is a circle of radius 3 (assume CCW).

SOLUTION: Since the curve is closed, if $\vec{F} = \langle P, Q \rangle$, then we see that $Q_x - P_y = 4$ and we can use Green's Theorem:

$$\iint\limits_{x^2 + y^2 \le 9} 4 \, dA = 4 \cdot \text{Area of circle} = 36\pi$$

(d) of the vector field $\vec{F} = \langle -y^2, x, z^2 \rangle$, and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ (C is CCW from above).

SOLUTION: Check the curl to see if the vector field is conservative: $\text{curl}(\vec{F}) = (1 + 2y)\vec{k}$. However, this does set us up to use Stokes' Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (1 + 2y)\vec{k} \cdot d\vec{S}$$

where the surface S is the plane $z = 2 - y$ above the circle $x^2 + y^2 = 1$. Therefore, the integrand is computed by taking the dot product:

$$\rangle 0, 0, 1 + 2y \rangle \cdot \langle -f_x, -f_y, 1 \rangle = \rangle 0, 0, 1 + 2y \rangle \cdot \langle 0, 1, 1 \rangle = 1 + 2y$$

Now do the integration:

$$\int_0^{2\pi} \int_0^1 (1 + 2r\sin(\theta))r \, dr \, d\theta = \cdots = \pi$$

ALTERNATIVE SOLUTION? We could have done this directly by parameterizing the circle with $x = \cos(t)$, $y = \sin(t)$, $z = 2 - \sin(t)$, but it would have been very time consuming!

24. A region E is a tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 2)$, $(0, 1, 0)$ and $(1, 1/2, 0)$.

(a) Find the three planes representing the three faces of E.
SOLUTION: Wasn't that a movie title? I think there are four faces... After a sketch, we see that the yz−plane is one face ($x = 0$), the xy−plane is another face ($z = 0$), the plane $y = \frac{1}{2}x$ is the third face, and the plane giving the top of the surface can be found by taking the cross product of two vectors on the surface. We found the normal vector to be $\rangle 1, 2, 1 \rangle$ (You can scale it if you like). So the equation of the plane is (using the point $(0, 0, 2)$:

$$x + 2y + (z - 2) = 0 \quad \Rightarrow \quad z = 2 - x - 2y$$

(b) Find six integrals that would give the volume of E. (NOTE: Careful in looking at the projection into the yz plane- there are actually two regions to consider).
SOLUTION:

i. If we use z for the height, z ranges from $z = 0$ to the plane $z = 2 - x - 2y$. In the xy−plane, we integrate over the triangle bounded by $x = 0$, $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x + 1$. From this, we get two possible integrals:

$$\int_0^1 \int_{y=1/2x}^{y=-1/2x+1} \int_0^{2-x-2y} dz \, dy \, dx$$

Or, we have to add:

$$\int_0^{1/2} \int_{x=0}^{x=2y} \int_0^{2-x-2y} dz \, dx \, dy + \int_{1/2}^1 \int_{x=0}^{x=-2y+2} \int_0^{2-x-2y} dz \, dx \, dy$$

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ii. If we use y for the "height", we project onto the xz plane. The variable y will range from the plane $y = \frac{1}{2}x$ to the plane $y = \frac{2-x-z}{2}$. The intersection of these planes forms the boundary line in the xz plane: $\frac{1}{2}x = \frac{2-x-z}{2}$, or $z = 2 - 2x$:

$$\int_0^1 \int_{z=0}^{z=2-2x} \int_{1/2x}^{(2-x-z)/2} dy\, dz\, dx$$

$$\int_0^2 \int_{x=0}^{x=(2-z)/2} \int_{1/2x}^{(2-x-z)/2} dy\, dx\, dz$$

iii. (See Figure 1). In the third case, we use x for the "height" over the yz plane. For some points in the yz plane, x will range from $x = 0$ to the plane $x = 2y$, but for other points, x will range from 0 to the plane $x = 2 - z - 2y$. The top part of the boundary in the yz plane is $z = 2 - 2y$, and the projection of the edge between the two planes is found by setting them equal to each other and removing the x component: $x = 2y$ and $x = 2 - z - 2y$ implies the intersection is $4y = 2 - z$. The more natural integral will be x, then y, then z. If we reverse z and y, we'll need three integrals:

$$\int_0^2 \int_{y=0}^{y=(2-z)/4} \int_{x=0}^{2y} dx\, dy\, dz + \int_0^2 \int_{y=(2-z)/4}^{(2-z)/2} \int_{x=0}^{2-z-2y} dx\, dy\, dz$$

or

$$\int_0^{1/2} \int_{z=0}^{2-4y} \int_0^{2y} dx\, dz\, dy + \int_0^{1/2} \int_{z=2-4y}^{2-2y} \int_0^{2-z-2y} dx\, dz\, dy+$$

$$\int_{1/2}^1 \int_{z=0}^{2-2y} \int_0^{2-z-2y} dx\, dz\, dy$$

25. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y) = x^2 y$ subject to the constraint $x^2 + 2y^2 = 6$.

SOLUTION: Recall that the candidate points are where the gradients of f and g (where $g(x,y) = k$ is the constraint) are parallel: $\nabla f = \nabla g \lambda$. This, together with the constraint $g(x,y) = k$ gives us the right number of equations for the number of variables we have. In this case, we have:

$$\begin{aligned} 2xy &= 2x\lambda \\ x^2 &= 4y\lambda \\ x^2 + 2y^2 &= 6 \end{aligned}$$

To solve this, go in a logical order. For example, from the first equation we see that either $x = 0$ or $\lambda = y$. Take each case and go to the second equation.

If $x = 0$ from the first equation, then $y = 0$ or $\lambda = 0$ in the second. But both x and y cannot be zero, since that would violate the third equation. Therefore, $\lambda = 0$, and $0^2 + 2y^2 = 6$, giving us points $(0, \pm\sqrt{3})$.
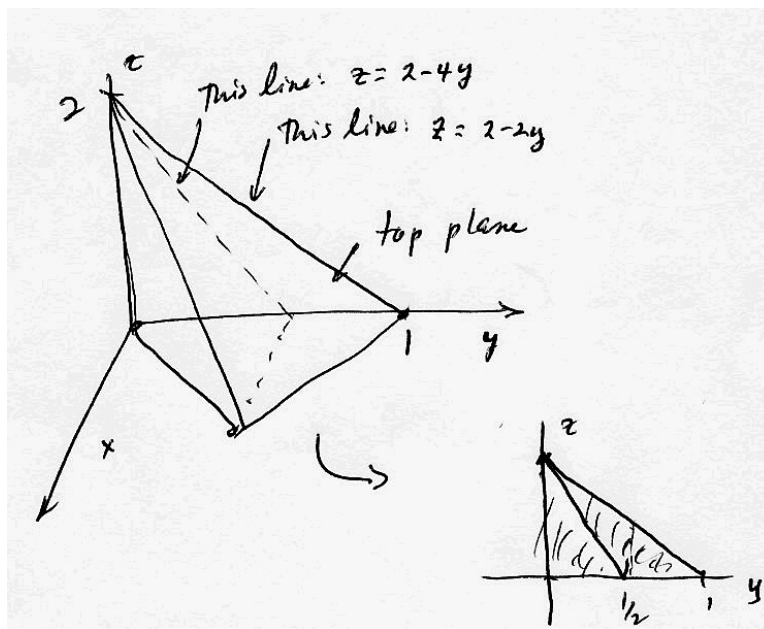
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Figure 1: Figure for the exercise in changing integration bounds.

If $\lambda = y$ from the first equation, then $x^2 = 4y^2$ in the second, and substituting that into the third equation, we get $6y^2 = 6$, or $y = \pm 1$. If $y = 1$, then $x = \pm 2$ (and same for -1). We get the four points: $(\pm 2, 1)$ and $(\pm 2, -1)$.

Put the critical points into f and find the largest and smallest:

$$f(0 \pm \sqrt{3}) = 0 \qquad f(\pm 2, 1) = 4 \qquad f(\pm 2, -1) = -4$$

Therefore, the maximum is 4 and the minimum is -4.

26. Set up an integral to determine the arc length of one period of the sine function (do not evaluate).

If we write the sine function as $y = f(x)$, we get:

$$\int_0^{2\pi} \sqrt{1 + \cos^2(x)}\, dx$$

(If you're curious, there is no elementary antiderivative, but numerically the arc length is approximately 7.64)

27. Use Stokes' Theorem to find the flux of the curl of $\mathbf{F}$ through the surface S, if

$$\mathbf{F} = \langle xz, yz, xy \rangle$$

and surface S is the part of the sphere $x^2 + y^2 + z^2 = 4$ inside the cylinder $x^2 + y^2 = 1$ and above the xy plane.

SOLUTION: 0 (See text, Example 2, 16.8)

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28. Use Green's Theorem to evaluate $\int_C x^2 y\, dx - xy^2\, dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

$$\iint_{x^2+y^2\leq 4} -(x^2 + y^2)\, dA = -16\pi$$

29. Find the flux across the surface:

$$\vec{F} = \langle xy, yz, zx \rangle$$

where the surface is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1,\, 0 \leq y \leq 1$.

SOLUTION: We won't be able to get this into a natural candidate for the Divergence Theorem, so just compute the integrand:

$$\vec{F}\cdot d\vec{S} = \langle xy, yz, xz \rangle \cdot \langle -f_x, -f_y, 1 \rangle\, dA = \langle xy, yz, xz \rangle \cdot \langle 2x, 2y, 1 \rangle\, dA = 2x^2 y + 2y^2 z + xz\, dA$$

Now we integrate over the square, substituting for z:

$$\int_0^1 \int_0^1 2x^2 y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2)\, dx\, dy$$

Don't spend a lot of time integrating this- if you have a lot of time to spare, the answer is $713/180$, or approximately 3.961.

30. Look over graphical problems: p. 1107, 1; p. 1104, 19; p. 1068, 9-11; p. 1053, 11; p. 1044, 17-18; p. 999, 33; p. 940, 1; p. 930, 3-4; p. 890, 70 (a-c); p. 889, 5-7.