Math 235: Calculus Lab

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Whitman College

Week 9


End Here
Start Here

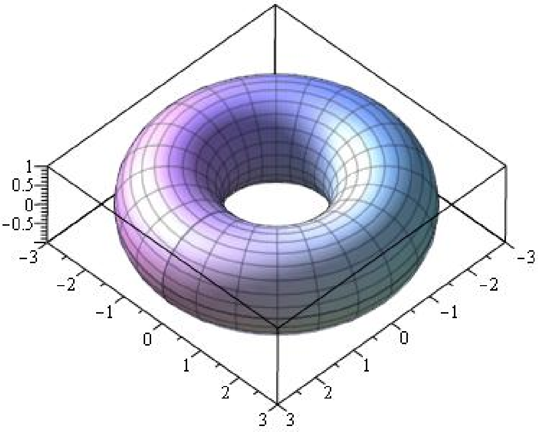
Ants and Three Paths: The goal is to go from $(3, 0, 0)$ to $(-3, 0, 0)$ on a path with the shortest distance.

Trial Runs:

1. Go along the “equator” of the doughnut.
2. Go directly to the inside circle, then go around, then climb back out.
3. Don’t go directly to the inside circle- Instead, go around both circles simultaneously.

The Torus

The torus is an object that looks like a doughnut:

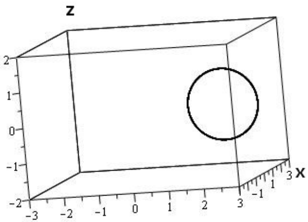


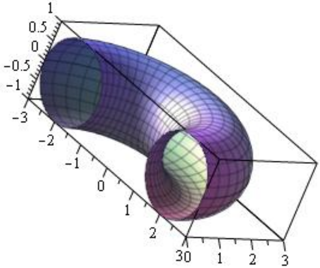
Torus Construction

Our torus is built by taking the graph of the unit circle:

$$(x-2)^2 + z^2 = 1$$

and spinning it around the $z-$axis:





Circles

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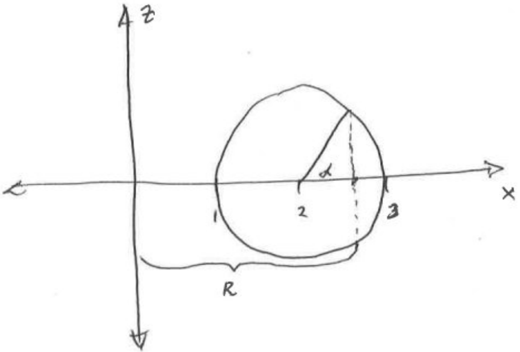
That is, for any point on circle of radius K, we can express that point as:

$$\begin{aligned} x(\theta) &= K\cos(\theta) \\ y(\theta) &= K\sin(\theta) \end{aligned}$$

Obtaining a Point on the Torus

To obtain any point on the surface of the torus we will:

- Start on the circle $(x-2)^2 + z^2 = 1$, and rotate through an angle α.



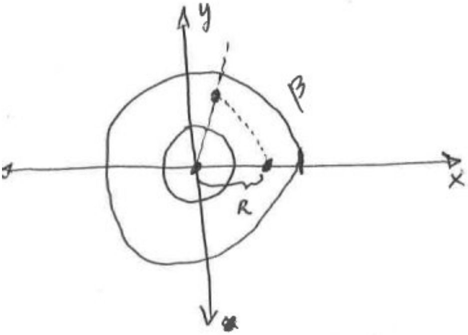
So far, before we rotate into the $xy-$plane,

$$R = \cos(\alpha) + 2$$

$$z = \sin(\alpha)$$

We rotate to get the x and y coordinates...

- In the $xy-$plane, we will then rotate through an angle β.



$$x = R\cos(\beta)$$

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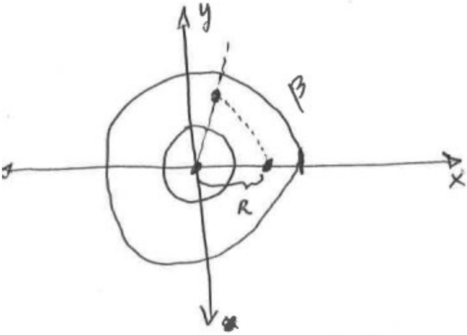

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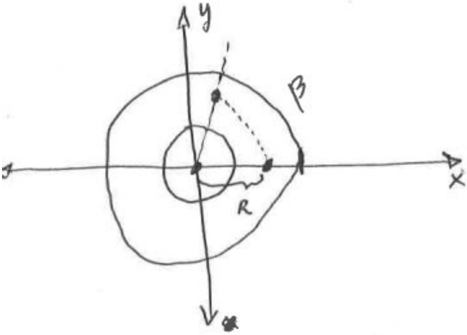

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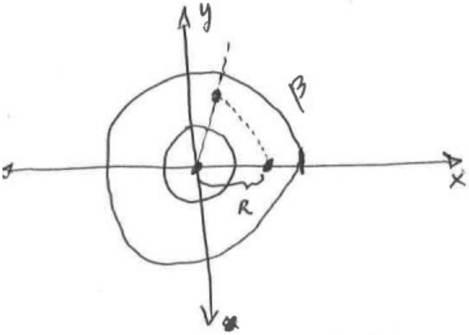

$$x = R\cos(\beta) = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = R\sin(\beta) = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha) \qquad \text{unchanged}$$

Conclusion thus far:

The surface of the torus can be expressed as:

$$x = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha)$$

Example points:

$$\beta = 0, \alpha = 0 \quad \Rightarrow$$

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$$\beta = \pi, \alpha = 0 \quad \Rightarrow \quad (-3, 0, 0)$$

Curves in the (β, α) plane:

If $\beta = \beta(t)$ and $\alpha = \alpha(t)$, then substituting these into

$$x = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha)$$

Creates the curve $\langle x(t), y(t), z(t) \rangle$ on the surface.

Path 1: Around the "Equator"

Path 1 keeps $\alpha = 0$ and β ranging from 0 to π. Therefore:

$$\begin{aligned}\beta(t) &= \pi t \\ \alpha(t) &= 0\end{aligned} \qquad 0 \leq t \leq 1 \qquad \Rightarrow \qquad \begin{aligned}x(t) &= 3\cos(\pi t) \\ y(t) &= 3\sin(\pi t) \\ z(t) &= 0\end{aligned}$$

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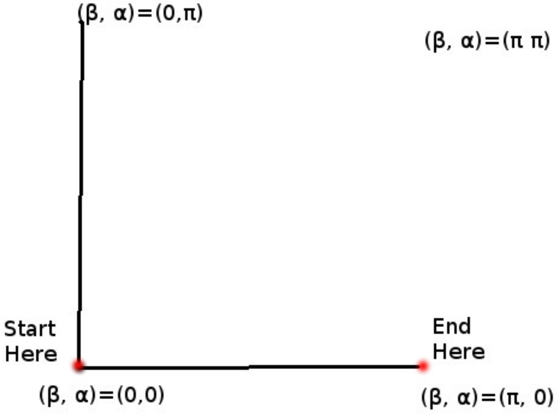
The (β, α) plane



Path 2

Path 2 is actually 3 paths:

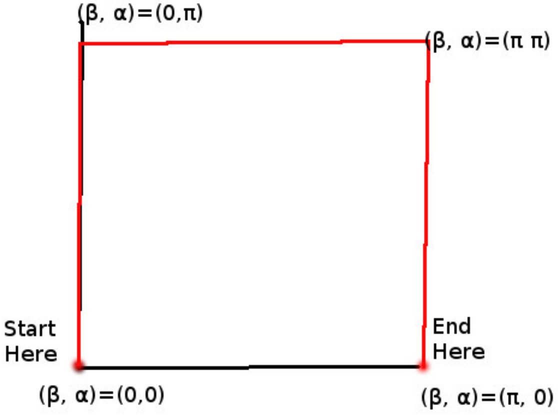
$$(0,0) \to (0,\pi) \to (\pi,\pi) \to (\pi,0)$$

Path 2A: $\beta = 0$, $\alpha = \pi t$
Path 2B: $\beta = \pi t$, $\alpha = \pi$
Path 2C: $\beta = \pi$, $\alpha = \pi(1 - t)$

In Maple, do these separately, and plot them all together.

Path Length: $\pi + \pi + \pi = 3\pi$

Path 2 in (β, α) plane



Path 3

In this case, take β from 0 to π. Then α will go from 0 to 2π.

$$\begin{aligned} \beta(t) &= \pi t \\ \alpha(t) &= 2\pi t \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(t) &= \cos(\pi t)(\cos(2\pi t) + 2) \\ y(t) &= \sin(\pi t)(\cos(2\pi t) + 2) \\ z(t) &= \sin(2\pi t) \end{aligned}$$

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Next Week: Continue with the current project and start to write results.