Lab 2: Building Blocks for Functions

1. Introduction to the Lab

In the pre-lab, we discussed the Taylor expansion for a function at a point, $x = a$ as:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

We can view this as a decomposition of $f(x)$ into pieces, or building blocks. That is, the building blocks are a constant, a line, a parabola, a cubic, etc. We construct functions by putting the blocks together in a certain way. This begs the question: Is there another way to construct a function using building blocks? Yes!

Suppose that we define

$$T_0(x) = 1,\ T_1(x) = x$$

and the remaining polynomials $T_k(x)$ are defined recursively:

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

so that, for example, $T_2(x) = 2xT_1(x) - T_0(1) = 2x^2 - 1$

Definition: The set of polynomials, $T_0, T_1, \dots$ are called *Chebyshev polynomials*.

We won't prove this, but if we have a function $f(x)$ defined on the interval $x \in [-1, 1]$, then we can write:

$$f(x) = a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + \dots$$

where the coefficients a_k are computed by:

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,dx,\ a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,dx \text{ for } k > 1$$

When you compute this in Maple, use -1.0 and 1.0 as the bounds- Maple will return numerical approximations to the integrals, which are fine.

2. The Lab Assignment

Compare and contrast the Maclaurin series and Chebyshev polynomials as methods for approximating a function f on the interval $-1 \le x \le 1$. As a starting point for your discussions, try to approximate $f(x) = e^x$ with either Maclaurin series up to degree k or Chebyshev polynomials up to degree k, and see which function is closer to e^x for $x \in [-1, 1]$ for each k. Try some different functions (remember that the approximation is only for $-1 \le x \le 1$) and see if the same thing happens- can you give some possible reasons for your findings?

You might find the following useful: In general, if $p(x)$ is an approximation to $f(x)$, then the error is given by $|p(x) - f(x)|$, and this can be plotted in Maple as: `plot(abs(p(x)-f(x)),x=-1..1);` You may estimate the maximum error off of the graph.

Inserting a Maple Plot Into Your Work

Before doing anything else, the line

`\usepackage{graphicx}`

should appear right below `\documentclass{amsart}`

Suppose I want to insert a figure of the graph of $\sin(3x)$. In Maple, go ahead and plot the figure. Click your mouse on the figure, and a black box should appear. Now if you RIGHT click the mouse in the black box, some menu items should appear. Select `Export As...`, then select `Encapsulated Postscript`. Now save your file- in this example, we saved the figure as `sine.eps`.

The latex code to insert this figure is shown below:

```
\begin{figure}[h]
\includegraphics[height=2.0in]{sine}
\caption{This is the graph of $y=\sin(3x)$}
\label{SineFig}
\end{figure}
```

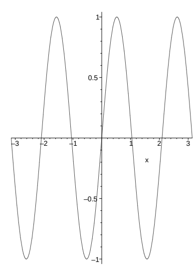

FIGURE 1. This is the graph of $y = \sin(3x)$

NOTES:

- Alternatives for the first line are: `\begin{figure}` or `\begin{figure}[h]` The differences are that in the first case, we are letting latex decide where the figure should go, and in the second case, we are suggesting to latex that the figure should go [h]ere.
- After `\includegraphics`, you can define how high or wide your image should appear. In the example, we defined the height to be 2 inches- This was only to save space- You would naturally not scale your images so small!
- In the same line, you don't need to give latex the full filename of your picture. It will assume that the filename ends with `.eps`
- Be sure to label your figure- you will reference it like you did equations. That is, the latex command `Figure \ref{SineFig}` will be realized as: Figure 1. Note that latex will number your figures for you.