# ARC LENGTH

DOUG HUNDLEY

## 1. Introduction

Let $y = f(x)$ be some function that is continuous on a closed interval, $[a, b]$. We wish to determine the length of the curve with represents the graph of $f$ on $[a, b]$.

We first subdivide the interval into $n$ equal pieces whose endpoints we denote by:

$$a = x_0, x_1, x_2, \ldots, x_n = b$$

We can now approximate the length of the curve by adding the length of the line segments starting at $(x_{i-1}, f(x_{i-1}))$ and ending at $(x, f(x_i))$, for $i = 1, 2, \ldots, n$.

By the Pythagorean Theorem, the length of the $i$th line segment is given by:

$$L_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \tag{1}$$

By the Mean Value Theorem, if $f$ is differentiable on $(a, b)$, then on the $i$th interval we can find $x_i^*$ so that:

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

Substituting this value into Equation 1, and writing $x_i - x_{i-1}$ as $\Delta x_i$, we have:

$$L_i = \sqrt{(1 + (f'(x_i))^2)\,\Delta x_i^2} = \sqrt{1 + (f'(x_i^*))^2}\,\Delta x_i$$

By summing all of the lengths, we get an approximation to the arc length. To get the exact value, we need to take the limit as the number of subdivisions gets very large:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(1 + (f'(x_i^*))^2)}\,\Delta x_i$$

This expression is the Riemann Sum for the integral, which gives the formula for the arc length of a curve. Here, we require that the function $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then the arc length is given by:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx \tag{2}$$

## 2. Example

Use Equation 2 to determine the arc length of the curve given by the graph of

$$y = x^{3/2}, \quad 1 \le x \le 4$$

We first differentiate to obtain $y' = \frac{3}{2}x^{1/2}$, so that $(y')^2 = \frac{9}{4}x$. Inserting this into Equation 2, we obtain the integral:

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\,dx$$

Now let $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}dx$, and

$$L = \frac{4}{9}\int_{13/4}^{10} \sqrt{u}\, du = \frac{8}{27}\left[10^{3/2} - (13/4)^{3/2}\right]$$