Solutions to the Wave Equation

1. Introduction

In this lab, we look at the wave equation, and look at different types of functions that solve it.

Definition: The wave equation is:

$$u_{tt} = a^2 u_{xx}$$

where u is a function of x and t. We say that a function *solves* the wave equation if we can substitute it for u, and the wave equation is satisfied.

2. Pre-Lab (We'll do some of these in-class)

Before you begin, let's explore some particular functions.

(1) Show that $u(x,t) = \sin(kx)\sin(akt)$ is a solution to the wave equation for all constants k. What is the effect of changing a and k? Try using `animate` and `plot3d` to figure it out.

(2) Show that $u(x,t) = \sin(x-at)+\ln(x+at)$ is a solution to the wave equation. What is the effect of changing the value of a?

(3) Show that, if f and g are any functions that are twice differentiable, then:

$$u(x,t) = f(x+at) - g(x-at)$$

solves the wave equation.

In fact, it can be shown that if our solution must satisfy the initial conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

(this is initial position and initial velocity) then the following function both solves the wave equation and satisfies the initial conditions:

$$u(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(v)\,dv$$

3. Lab

Write up your findings from the pre-lab. Try to incorporate them into a discussion, using these functions as examples. Come up with your own initial conditions and use Maple to find the solution- The group with the most complicated motion will get a special prize! If you include a 3-d plot, you might want to render it in greyscale before saving it, since we're printing in black and white.

Note that this lab is a bit different than the ones we've had before- I'm not asking you to solve a particular problem, so you may not have a "Conclusions" section. Think of this lab as a treatise on solutions to the wave equation, and the possible solutions you can get. You may incorporate any of the material on this page in any order that you want.

Side Remark: In the physical setting, this equation describes a vibrating string. The constant a refers to physical constants:

$$a = \sqrt{\frac{Tg}{w}}$$

where T is the tension, g is gravity, w is weight per unit length. In the physical setting, the ends of the string were clamped- but we can look at non-physical solutions as well!