

Which is Faster, Going Up or Coming Down?¹

Suppose you throw a ball into the air. Do you think it takes longer to reach its maximum height or to fall back to Earth from its maximum height? We will solve the problem in this project but, before getting started, think about that situation and make a guess based on your physical intuition.

1. A ball with mass m is projected vertically upward from Earth's surface with a positive initial velocity v_0 . We assume the forces acting on the ball are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude $p|v(t)|$, where p is a positive constant and $v(t)$ is the velocity of the ball at time t . In both the ascent and descent, the total force acting on the ball is $-pv - mg$. (During the ascent, $v(t)$ is positive and the resistance acts downward; during descent, $v(t)$ is negative and the resistance acts upward.) So, by Newton's Second Law, the equation of motion is

$$mv' = -pv - mg.$$

Solve this differential equation.

2. Determine $y(t)$, the height of the ball at time t (until it hits the ground).
3. Let t_1 be the time that the ball takes to reach its maximum height. Show that

$$t_1 = \frac{m}{p} \ln \left(\frac{mg + pv_0}{mg} \right).$$

Find this time for a ball with mass 1 kg and initial velocity 20 m/s. Assume the air resistance is 1/10 of the speed.

4. Let t_2 be the time at which the ball falls back to Earth. For the particular ball in Problem 3, estimate t_2 by using a graph of the height function $y(t)$. Which is faster, going up or coming down?
5. In general, it's not easy to find t_2 because it's impossible to solve the equation $y(t) = 0$ explicitly. We can, however, use an indirect method to determine whether ascent or descent is faster; we determine whether $y(2t_1)$ is positive or negative.

Show that

$$y(2t_1) = \frac{m^2 g}{p^2} \left(x - \frac{1}{x} - 2 \ln x \right)$$

where $x = \exp(pt_1/m)$. The show that $x > 1$ and the function

$$f(x) = x - \frac{1}{x} - 2 \ln x$$

is increasing for $x > 1$. Use this result to decide whether $y(2t_1)$ is positive or negative. What can you conclude? Is the ascent or descent faster?

¹taken from Calculus, 4th ed. Stewart