Lab 3: Clairaut's Theorem[1]

A famous theorem from Calculus is the theorem of Clairaut- That mixed second partial derivatives are equal. Because most functions we work with are "nice", it is easy to think that Clairaut's Theorem applies to *every* function- In this lab, we will see that it does not. Go through the questions in this lab, using Maple for limits and graphs. Try to incorporate the answers to the questions in a narrative form. The grading criteria are listed on our class website.

Theorem 0.1 (Clairaut)**.** *Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Consider the function

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

(1) First get a feeling for what f is by plotting it. You don't need to do a piecewise definition for the plot.

(2) Is f continuous at the origin?
   (a) Try using the definition (limits) in Maple.
   (b) Try using the squeeze theorem. (Hint: You might look to see if $-|xy| \leq f(x,y) \leq |xy|$. The absolute value function in Maple is `abs()`)

(3) Is f differentiable?
   (a) Compute f_x, f_y using Maple (You might also use `simplify` here).
   (b) Compute $f_x(0,0)$ and $f_y(0,0)$ by using the definition of the derivative.
   (c) Show that f_x and f_y are continuous at the origin by seeing that $-2|y| \leq f_x(x,y) \leq 2|y|$, and similarly, $-2|x| \leq f_y(x,y) \leq 2|x|$. Can you show these algebraically? Do they help prove continuity at the origin? (How?)

(4) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$. (Note: Compute $f_{xy}(0,0)$ using the *definition* of the derivative).

(5) Discuss why Clairaut's Theorem does not apply here.

[1]Calculus On Manifolds by M. Spivak, adaptation of exercise 24, page 29