Ants on a Doughnut[1]

   Two ants are exploring a doughnut and notice that there are two regions on the doughnut which have particularly good supplies of sugar and grease. The two regions are as far apart on the doughnut as they can be. The ants must leave a path for all the other ants to follow. In order to be efficient the ants wish to find the shortest path between the good regions.

   The ants notice that the doughnut is obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ (in the xz-plane) about the z-axis. This puts the doughnut in xyz-space. The areas rich in sugar and grease are at the positions $(3, 0, 0)$ and $(-3, 0, 0)$.

   (**Path 1:**) The first ant, Grant, suggests that the shortest route would be to follow the path of $(3, 0, 0)$ as it rotates about the z-axis as in Figure 1

   (**Path 2:**) Antonio, the second ant suggests that the "inside" circle is smaller, so it would make sense to follow the circle $(x - 2)^2 + z^2 = 1$ half way around so they are at the "small" circle, follow it half way around to the point $(-1, 0, 0)$, and then go around the circle from $(-1, 0, 0)$ to $(-3, 0, 0)$.

   (**Path 3:**) After hearing Antonio's idea Grant thinks that it would be better not to go directly around the circle $(x - 2)^2 + z^2 = 1$ before starting around the z-axis rotation. Grant says "Why not go around both circles at the same time? Perhaps we could go around both circles simultaneously and save some distance."

   (**Path 4:**) Antonio then suggests going around both at the same time part way until they hit the 'inside circle' on the xy-plane, around the inside circle for a ways and finally around both until they arrive at $(-3, 0, 0)$. Their path could be symmetric about the yz-plane.

   The Goal of this Lab: Try to find a shortest path between the two given points.

   To put this lab into a larger context, consider that typically we say that the shortest path between two points is a straight line- but it really depends on the geometry of the space. In this lab, our space is the surface of a torus- And in that case, what we find is that the shortest path is not a straight line.

   Here are some things to get started:

1. Think of each point on the torus (or doughnut) as being specified by two circles. One circle is given in the (x, z) plane, $(x - 2)^2 + z^2 = 1$. Let α measure the angle on this circle.

   The second circle is given by rotating the center of the previous circle around the (x, y) plane. Let β be the angle measure along this circle.

   Verify that the equations in α, β that give the torus are:

$$x(\alpha, \beta) = \cos(\beta) \left(\cos(\alpha) + 2 \right),$$
$$y(\alpha, \beta) = \sin(\beta) \left(\cos(\alpha) + 2 \right),$$
$$z(\alpha, \beta) = \sin(\alpha)$$

   See the sample Maple worksheet to plot the torus (doughnut).

2. (In Class) Make a graph of the paths 1 and 2 in the $\alpha\beta$-plane. Find the lengths of these paths (the length is measured on the surface of the torus).

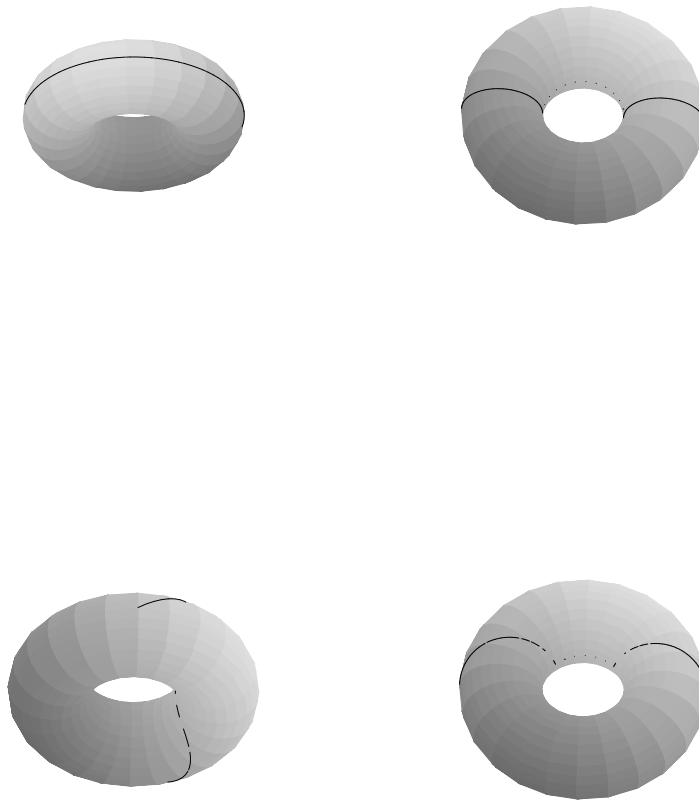
[1] Adopted from Neal Brand, University of North Texas

Figure 1: Staying on the outside of the dougnut. These are the 4 sample paths, from left to right, top to bottom.

3. (In Class) Repeat the previous problem using path 3. In this case, we might interpret Grant's suggestion as simultaneously moving α from 0 to 2π, while continuously moving β from 0 to π. In this case, $\alpha(t) = 2\pi t$, and $\beta(t) = \pi t$, where $0 \leq t \leq 1$.

   Try to change the sample worksheet to reflect this path and compute the path length. You may use `SamplePath3.mws` to compare.

4. (In Class) Repeat the previous problem using path 4. In this case, we interpret Antonio's suggestion as three distinct paths:

   - α goes from 0 to π while β goes from 0 to some value k. This gets us to the inside circle. In parametric form, $\alpha(t) = \pi t$ and $\beta(t) = kt$, $1 \leq t \leq 1$.
   - α is fixed at π, while β goes from $\pi/2 - k$ to $\pi/2 + k$. This moves us around the inside cirlce.

In parametric form,

$$\alpha(t) = \pi \qquad\qquad \beta(t) = (1-t)(\pi/2 - k) + t(\pi/2 + k) \qquad 0 \le t \le 1$$

- α moves from π to 0, while β moves from $\pi/2 + k$ to π. This gets us to $(-3, 0, 0)$.
  In parametric form,

$$\alpha(t) = (1-t)\pi \qquad\qquad \beta(t) = (1-t)(\pi/2 + k) + t\pi$$

In this case, the path length depends on k. We can use Maple to construct the path length function, solve for the best value of k, then plot the results.

See the sample Maple file, `SamplePath4.mws`