NOTES: Lab 3, Clairaut

Here are some notes to help you with the algebra in the lab:

1. Let
$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$$
Notice that this could be written as:
$$f(x,y) = xy \cdot \left(\frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} \right)$$
To show that $-|xy| \leq f(x,y) \leq |xy|$, show that the quantity in parenthesis is between ± 1:

   We note that the two numbers in parenthesis must sum to 1. We could rephrase that: Let $a+b=1$, where $0 \leq a, b \leq 1$. Find the max and min of $a-b$. We could do a substitution, so we would find the max and min of $a-(1-a) = 2a-1$, where a is between 0 and 1. The maximum is attained at $a=1$ ($b=0$) and the minimum is where $a=0$ ($b=1$). Therefore,
$$-1 \leq \frac{x^2-y^2}{x^2+y^2} \leq 1$$

2. Show that f_x should be 0 at $(x,y)=(0,0)$ by seeing that it is trapped between $2|y|$ and $-2|y|$. To show this, note that:
$$\frac{x^4-y^4+4x^2y^2}{(x^2+y^2)^2} = \frac{x^4+2x^2y^2+y^4+2x^2y^2-2y^4}{(x^2+y^2)^2}$$
We wrote the fraction in this way to simplify things a bit:
$$\frac{x^4+2x^2y^2+y^4+2x^2y^2-2y^4}{(x^2+y^2)^2} = \frac{(x^2+y^2)^2+2y^2(x^2-y^2)}{(x^2+y^2)^2}$$
Now simplify:
$$\frac{(x^2+y^2)^2+2y^2(x^2-y^2)}{(x^2+y^2)^2} = 1 + 2\frac{y^2}{x^2+y^2}\left(\frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} \right)$$

As before, let $a = x^2/(x^2+y^2)$, and $b = y^2/(x^2+y^2)$. Then $a, b \geq 0$, $a + b = 1$, and we want to find the minimum and maximum of:

$$1 + 2b(a - b)$$

Substituting $a = 1 - b$, we find the min and max of

$$1 + 2b((1-b) - b) = 1 + 2b(1-2b) = 1 + 2b - 4b^2, \qquad 0 \leq b \leq 1$$

Using Calculus, you should find the the maximum occurs at $b = 1/4$, and the minimum occurs at $b = 1$. Put these back into the expression to see that:

$$-1 \leq \frac{x^4 - y^4 + 4x^2y^2}{(x^2+y^2)^2} \leq \frac{5}{4}$$

Therefore,

$$-|y| \leq f_x(x,y) \leq \frac{5}{4}|y|$$

Now see if you can do something similar for $f_y(x,y)$.

3. For the second mixed partials, try plotting. Does the graph look familiar (like something from our practice Maple sheet)?

   Algebraically, take note of $f_x(0,y)$ and $f_y(x,0)$. Then compute:

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0, 0+h) - f_x(0,0)}{h}$$

   and

$$f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(0+h, 0) - f_y(0,0)}{h}$$

Some things I want you to get from doing Lab 3:

- Maple is a very powerful visualization and computational tool, especially in three dimensions.
- We should never trust Maple completely- Always do a "reality check" on what Maple is giving you to see if you believe it.
- From our algebra in these notes, it's clear that while Maple is a great tool, nothing beats old fashioned mathematical reasoning and skill!