Lab Shortcuts

Since we're running a little short on time this semester, here is a simplified version of our lab. Our lab is dealing with the *isoperimetric problem*, which is:

If we want to maximize the integral

$$\frac{1}{2}\int_0^1 y\dot{x} - x\dot{y}\,dt$$

subject to the constraint that:

$$\int \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = k$$

for some constant k, then we can construct a function H (from the Lagrange Multipliers) as:

$$H = \frac{1}{2}(y\dot{x} - x\dot{y}) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}$$

It turns out that, for our functions x, y to be optimal, the following equations must be true (these are called the Euler-Lagrange Equations):

$$\frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0$$

and

$$\frac{\partial H}{\partial y} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{y}}\right) = 0$$

- Show that the first Euler Lagrange equation simplifies to:

  $$\frac{d}{dt}\left[y + \frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] = 0$$

  so that, for some arbitrary constant C_1,

  $$y - C_1 = -\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

- Similarly, the second equation gives:

  $$\frac{d}{dt}\left[x - \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right] = 0 \quad \Rightarrow \quad x - C_2 = \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

- Finally, show that your previous two problems simplify to the equation of a circle:

  $$(x - C_1)^2 + (y - C_2)^2 = \lambda^2$$

  Thus, the curve that gives the optimal area is a circle.