Isolating Perry Metric in a Quality Cell

Purpose: The purpose of this project is to introduce you to a type of optimization problem and solution procedure which is based on multivariable calculus.

Procedure: You should follow the steps below (some parts need previous results), and as is our usual practice, write up your results in a narrative form. In this last lab, you may decide on the format.

Perry Metric was caught stealing a balloon from a toy store. Perry's lawyer plea bargained, and was able to reduce Perry's sentence to two years in a cell designed by Perry. The judge told Perry that he could design the cell to be any shape that Perry wanted as long as the perimeter was no more than 40 meters. Perry's goal is to enclose as much area as possible and have no corner in his cell wall. Perry is pretty sure what shape the wall should be in order to accomplish his goal. After thinking about the problem he decides that he had better be safe and see if there is justification for his conjecture concerning the shape of the wall. Perry calls you for help in proving his conjecture. The judge is only giving Perry two weeks to design the cell, so he needs a proof within two weeks or else the judge will impose the shape of the cell (which will probably be a 1 meter by 19 meter rectangle). Before calling you, Perry looked through some mathematics books and found a few theorems which may be of use to you.

Theorem 1: Suppose that $F(x, y, t)$ is a smooth function. Then

$$\frac{\partial}{\partial x}\int_a^b F(x,y,t)dt = \int_a^b \frac{\partial}{\partial x}F(x,y,t)dt$$

Actually, the above theorem is true if F is a function of more than three variable as well. You should interpret $\int_a^b F(x,y,t)dt$ as being a function of two variables, x and y. So it makes sense to take the partial derivative of this function with respect to x, or y.

The next theorem is a special case of Green's theorem:

Theorem 2: Suppose that a planar curve is given in parametric form by smooth functions $x(t)$ and $y(t)$ for $0 \le t \le 1$. Furthermore, suppose that $x(0) = x(1)$ and $y(0) = y(1)$, but otherwise the curve does not intersect itself (a simple closed curve). Then the area enclosed by the curve is given by

$$\frac{1}{2}\int_0^1 xy' - yx'dt$$

as long as the curve goes around the area in a counter–clockwise direction.

The third theorem is useful at times when you want to show that a function is 0 for every point in its domain.

Theorem 3: If $g(x)$ is continuous and $\int_0^1 g(x)\eta(x)dx = 0$ for every smooth function $\eta(x)$ having domain $[0, 1]$ with $\eta(0) = \eta(1) = 0$ then $g(x) = 0$ for every $0 \le x \le 1$.

It may not be obvious how to solve Perry Metric's problem using the three theorems listed, but hopefully with the outline below you will be able to do it. The first three parts are to help you understand the theorems which you will need to solve Perry's problem.

- Do a few examples of Theorem 1. Make up your own functions for F where you can do the indicated integrals (polynomials work well) and check that both sides are the same.
- Next write the parametric equations for an ellipse in standard form and use Theorem 2 to compute the area inside the ellipse. Compute the area of the ellipse in another way to make sure you obtain the correct answer. What happens if the curve goes around the area in a clockwise direction?
- Draw a picture and explain why you think Theorem 3 should be true. (Hint: suppose that $g(x) > 0$ at some point. Draw a picture of a function $\eta(x)$ which would make $\int_0^1 g(x)\eta(x)dx > 0$.)

Now that you know what the theorems are saying and perhaps have a feeling about why they are true you may use the theorems to complete the steps below.

(1) Suppose that the wall of Perry's cell is given by a pair of parametric equations which are parameterized by t with $0 \leq t \leq 1$. (The parametric equations give the curve on the floor where Perry's wall touches the floor.) We may as well assume that $x(0) = y(0) = 0 = x(1) = y(1)$, that is, the curve starts and ends at the origin since the area and arc length is not changed if the curve is translated to a different position in the plane. Write formulas which give the arc length and the area enclosed by Perry's walls. Keep in mind that x, y, x', and y' are functions of t, and not simply variable.

(2) Suppose that $x(t)$ and $y(t)$ define a parametric curve giving Perry's desired wall. Then for an arbitrary (but fixed) pair of smooth functions $\eta_1(t)$ and $\eta_2(t)$ and arbitrary values of the variables z and w, write conditions under which the curve parameterized by $z\eta_1(t) + x(t)$ as the x-value and $w\eta_2(t) + y(t)$ as the y-value is a closed curve of length 40 which starts and ends at the origin. Also write the formula for the area enclosed by this curve.

(3) Now, if $(x(t), y(t))$ give the desired wall for Perry then the values $w = 0$ and $z = 0$ should yield a maximum for the area formula you found in part 2 subject to the conditions you gave in part 2. Write the Lagrange multiplier formula for this situation. (You will need to use Theorem 1 and the chain rule.)

(4) Simplify your previous answer as much as you can. Try to use Theorem 3 to conclude that a certain function is zero. (Hint: it may be useful to use integration by parts on two of the integrals in each formula you derive and then combine the integral formulas in order to use Theorem 3.)

(5) Can you now conclude that the curve is what you suspect it should be? (With some work, the answer should be yes.)

(6) Think carefully about what the above calculations show and give a clear statement of what you proved.