THE BICYCLE RACE

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1. INTRODUCTION

We will consider the situation of a cyclist passing a refreshment station in a bicycle race and the relative positions of the cyclist and her chasing support car. The cyclist rides at a constant velocity with her distance in meters from the refreshment station given by kt where t is time in seconds after passing the station. The pursuing support car's distance in meters from the station is given by $\frac{1}{3}(10t^2 - t^3)$. There are several aspects of the situation we will analyze.

We are given the position of the cyclist relative to the location of the refreshment station as a function of time. We denote by $s_c(t)$ the position of the cyclist at time t. Denote by $s_a(t)$ the position function of the auto. From the discussion in §2.3 of Calculus, Stewart 3rd ed, we know that the velocity of the cyclist is $s_c'(t)$ and the velocity of the auto is $s_a'(t)$. In §2 of this article we compute the cyclist's velocity and the time at which the chase car catches the cyclist. In §3 we explore the effect of different cyclist velocities on meeting times. In §4 we attempt to illustrate the various scenarios graphically. Finally, in §5 we connect a theortical result about cubic polynomials with the situation of the cyclist and the chase car.

2. VELOCITY AND MEETING TIME

In this section we determine the speed of the cyclist and how long it takes the chase car to catch her. We have the additional information that when the chase car catches the cyclist their velocities are the same. This provides us with two algebraic equations. When the chase car catches the cyclist their positions are the same, thus

$$s_c(t) = s_a(t).$$

Furthermore, when their positions coincide, by design, their velocities are the same, thus

$$s_c'(t) = s_a'(t).$$

Together, we have two equations in the two unknowns k and t.

$$kt = \frac{1}{3}(10t^2 - t^3) \tag{1}$$

$$k = \frac{1}{3}(20t - 3t^2) \tag{2}$$

Using the equation (2), we eliminate k from the equation (1) and solve for t. We find two solutions, $t = 0$s and $t = 5$s. The first solution is consistent with the fact that their positions coincided at the refreshment station. The second time, 5 seconds, is when the support car catches the cyclist.

The velocity of the cyclist as a function of time is given by $s_c'(t) = k$, where k is a constant. Determining k will give the velocity of the cyclist. We know that at $t = 5$s equation (2) holds. We can determine k by substituting $t = 5$s into equation (2). Doing so we find $k = 25/3$ m/s. Thus the cyclist is traveling at a rate of $25/3$ m/s.

3. DIFFERENT VELOCITIES

We will now consider what happens if k is allowed to vary. We will derive a formula for t in terms of k for the times when the position of the cyclist and the car coincide.

By setting the two position functions equal to each other and solving for t, we can determine the length of time it takes for the pursuing car to catch the cyclist.

$$\begin{aligned}
s_c(t) &= s_a(t) \\
kt &= \frac{1}{3}(10t^2 - t^3) \\
3kt &= 10t^2 - t^3 \\
3k &= 10t - t^2, \ t \neq 0 \\
t^2 - 10t + 3k &= 0, \ t \neq 0
\end{aligned}$$

From this calculation we see that $t = 0$ is one solution, which is consistent with the fact that their positions are the same when the cyclist passes the refreshment station. We see that there are potentially two more times where the car and the cyclist have the same position. To determine these times, we must apply the quadratic formula,

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 1$, $b = -10$ and $c = 3k$. Doing so we have

$$t = \frac{10 \pm \sqrt{100 - 12k}}{2}.$$

In order for the solutions to be real–valued, we must have $k \leq 25/3$. Assuming $k > 0$, we see that the expression for t is always positive and yields two distinct times. From this we may conclude that if she rides faster than $25/3$ m/s, the support car will not catch her. If she rides slower than $25/3$ m/s, the support car will meet her twice.

4. Graphical illustration

In this section we provide graphs to illustrate what happens in the first case where the car and the cyclist meet and have the same velocity and in the second case where the velocity may change. Figure 1, shows the position functions of the cyclist and the car on the same axes with $k = 25/3$ m/s–the optimal speed. We see that after $t = 0$s the two functions intersect only once at $t = 5$s.

FIGURE 1. $k = \frac{25}{3}$, Cyclist–Solid Line, Auto–Dotted Line.

Figure 2 shows the same graph with the cyclist's velocity slightly faster at $k = 26/3$ m/s. In this case we see that the two plots fail to intersect after $t = 0$s.

FIGURE 2. $k = \frac{26}{3}$, Cyclist–Solid Line, Auto–Dotted Line.

Figure 3 shows the same graph with the cyclist's velocity slightly slower at $k = 20/3$ m/s. In this case we see that the two plots intersect twice after $t = 0$s.

FIGURE 3. $k = \frac{20}{3}$, Cyclist–Solid Line, Auto–Dotted Line.

In the situation depicted in Figure 2, it would be difficult for the support car to give refreshment since there would still be some distance between the car the cyclist.

In the situation depicted in Figure 3, it would be difficult for the car to give refreshment because the velocities of the cyclist and the car would be too different.

5. Cubic polynomials

We will now show that if a cubic polynomial, $P(t)$, has a repeated real root at $t = a$, then $P'(a) = 0$. We will then use this result to interpret the relationship between the cyclist and the auto. Let $t = a$ be a double root. If b is the remaining root of $P(t)$, then we may write

$$P(t) = K(t-a)^2(t-b)$$

where K is the coefficient on the t^3 term. Thus,

$$P'(t) = K(t-a)^2 + 2K(t-a)(t-b).$$

Hence, $P'(a) = 0$.

This result is relevant to the first situation in which the car and the cyclist are traveling at the same speed when they meet. The cubic polynomial in question is $P(t) = s_c(t) - s_a(t)$ and represents the difference in the positions of the cyclist and the car. Using the **factor()** call in maple with $k = 25/3$ m/s we have

$$P(t) = \frac{1}{3}t(t-5)^2.$$

We see that $t = 5$ is a double root of P. Notice further that P' gives the difference in the velocities of the cyclist and the auto. The above double root result states that $P'(5) = 0$ which is consistent with the fact that at the time that they meet the cyclist and the auto are traveling at the same speed.

In the later examples where k is allowed to vary, we find that P has not double roots and the delicate condition of matching position and speed is destroyed.