Newton’s Method in 2 Dimensions

The ordinary Newton’s Method uses the linear approximation to find an approximate solution to an equation of the form \( f(x) = 0 \). Basically, if \( x_0 \) is an initial approximation to the solution, then the tangent line to \( y = f(x) \) at \( x = x_0 \) intersects the \( x \)-axis at a point \((x_1, 0)\), and \( x_1 \) is usually a better approximation to the solution than \( x_0 \). So the process can be iterated using \( x_1 \), and a short derivation shows that at each stage,

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

This may be automated in Maple by defining the function

\[
> \text{newt} := x \rightarrow \text{evalf}(x-f(x)/Df(x));
\]

Here is a full example. We will plot the curve to estimate a starting value \( x_0 \). Here, after viewing the plot, we decide on \( x_0 = 0.8 \). (Note about typing the for loop: Use the shift then enter keys to get a new line without Maple evaluating the expression).

\[
> \text{Digits} := 16;
> f := x \rightarrow \cos(x)-x;
> Df := D(f);
> \text{newt} := x \rightarrow \text{evalf}(x-f(x)/Df(x));
> \text{plot}(f(x), x=-\Pi..\Pi);
> t := 1.2;
> \text{for i from 1 to 4 do}
> \quad y := \text{newt}(t);
> \quad \text{if abs}(y-t)<10^{-8}
> \quad \text{then printf(”Done on iterate %d”, i);
> \quad \quad printf(” and the solution is %f\n”, y);
> \quad \quad break;
> \quad \text{else}
> \quad \quad t := y;
> \quad \end{\text{if}}
> \end{\text{for}};
\]

\textit{NOTES:} New in Maple- The \( D \) command, and look over the ”for loop”, which is ended by the \( \text{od} \) line. There is also an if statement.

\footnote{Adopted from \textit{CalcLabs with Maple for Stewart’s Multivariate Calculus}, P.B. Yasskin and A. Belmonte}
Two dimensional Newton’s Method

The two dimensional Newton’s Method works in the same way- We are looking for an ordered pair \((x, y)\) that solves the pair of equations

\[
f(x, y) = 0 \quad g(x, y) = 0
\]

If \((x_0, y_0)\) is an initial approximation to the solution, then the tangent plane to \(z = f(x, y)\) at \((x_0, y_0)\) and the tangent plane to \(z = g(x, y)\) at \((x_0, y_0)\) intersect the \(xy\)-plane at a common point, \((x_1, y_1, 0)\). Hopefully \((x_1, y_1)\) is a better approximation to the solution than \((x_0, y_0)\).

Lab Questions

1. Derive equations for \(x_{i+1}\) and \(y_{i+1}\) like we had for the original Newton’s Method. Your solutions will depend on \(f, f_x, f_y, g, g_x, g_y\), all evaluated at \((x_i, y_i)\). Your answer should be in the form:

\[
x_{i+1} = x_i - \quad \quad \quad y_{i+1} = y_i - \quad \quad \quad
\]

2. Construct a single Maple function, \texttt{newt2d} which acts on an initial approximation and produces the next approximation. Here is a simple example of a function that takes in two numbers and produces two numbers for iteration:

\[
> \text{newtex:}=(x,y)->(3*x-5*y, 2*x+y);
> g:=newtex(1,2);
> \text{newtex}(g);
\]

3. Use your Maple function to find all solutions to each of the following pairs of equations. You will need to plot the two equations using \texttt{implicitplot} to get an initial approximation to each solution. Iterate enough so that the maximum difference between two successive iterations is no more than \(10^{-10}\). You can use \texttt{fsolve} to check your solutions.

(a) \(x + y - \cos(x) + \sin(y - 1) = 0\) and \(x^4 + y^4 - 2xy = 0\)
(b) \(5x - 3y = -2\) and \(2x - 2y = -3\)
(c) \(y^3x - x^3y + x^2y^2 = 7\) and \(2x^4 + 3y^4 = 74\)