Lab 2: *Clairaut's Theorem*

A famous theorem is that the mixed partial derivatives of certain nice functions are the same- This is Clairaut's Theorem.

Because most functions we work with are nice, it is easy to think that Clairaut's Theorem applies to *every* function- In this lab, we will see that it does not. Go through the questions in this lab, using Maple for limits and graphs. Try to incorporate the answers to the questions in a narrative form. The grading criteria are listed on our class website.

Goals for this Lab

- Be able to graphically determine if a surface is continuous and/or differentiable.
- Be able to compute the partial derivatives using Maple and (by hand) using the definition.
- Understand how a given function that fails the hypotheses of Clairaut's Theorem may NOT have equal mixed partials.
- For this lab, you are writing up one particular example in detail. To get you started, think about how you would describe the problem to another student in class.
- It is easiest to write it up if you already know what the answers are, so try to answer the outline questions below before you get started. Then think about how you can put the answers into a narrative.
- Grading will be based on the following (5 pts each):
  - Put into narrative form. The structure of the paper follows a logical path to the conclusions. Figures are used to illustrate the ideas.
  - Spelling, grammar, punctuation and typesetting (like parentheses, equation labeling, etc)
  - The mathematics are clearly explained, correct and complete.

Lab 1 Details

The overall goal of this lab is, given the particular function below, show that it does not satisfy the conditions of Clairaut's Theorem. To assist you, there are several items below that will guide your thinking. In particular, you might first think about whether the function is continuous. Then look at if it is differentiable, and twice differentiable. The theorem is included below for your convenience.

Theorem (Clairaut)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Consider the function

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

1. First get a feeling for what f is by plotting it (for the plot, Maple will ignore the origin so you can ignore the possible point of discontinuity at the origin).
2. Is f continuous at the origin?
   (a) Try using the definition (limits) in Maple.
   (b) Try using the squeeze theorem. (Hint: You might look to see if $-|xy| \leq f(x,y) \leq |xy|$. The absolute value function in Maple is `abs()`)
3. Is f differentiable?
   (a) Compute f_x, f_y using Maple (You might also use `simplify` here).
   (b) Compute $f_x(0,0)$ and $f_y(0,0)$ by using the definition of the derivative (by hand).
   (c) Show that f_x and f_y are continuous at the origin by seeing that $-2|y| \leq f_x(x,y) \leq 2|y|$, and similarly, $-2|x| \leq f_y(x,y) \leq 2|x|$. Can you show these algebraically? Do they help prove continuity at the origin? (How?)
4. Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$. (Note: Compute $f_{xy}(0,0)$ using the *definition* of the derivative).
5. Discuss why Clairaut's Theorem does not apply here.

Some helpful definitions:

- The function $z = f(x,y)$ is continuous at $x = a, y = b$ if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

- Given $z = f(x,y)$, the partial derivative with respect to x at the point $x = a, y = b$ is:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

  Similarly, the partial derivative with respect to y:

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$