Just as a parametric curve can be thought of as a mapping of the real line to either 2 or 3 dimensions:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

a **parametric surface** is a mapping of two dimensions (usually denoted by (u, v)) into three (or more) dimensions:

$$\mathbf{F}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

The special case is the one we're accustomed to: Let $u = x$, $v = y$, and $z = f(x, y)$, and the mapping is:

$$\mathbf{F}(x, y) = \langle x, y, f(x, y) \rangle$$

Think of a point in parameter space, (u, v), as if it were a point on a bed sheet. Then the triple

$$\langle x(u, v), y(u, v), z(u, v) \rangle$$

tells you the 3 dimensional coordinates of that point. Therefore, a "surface" is defined just as you might drape a bed sheet in three dimensions.

We can plot surfaces parametrically in Maple:

```
plot3d([f,g,h],u=a..b,v=c..d);
```

where $x = f(u, v)$, $y = g(u, v)$ and $z = h(u, v)$- For example, we could plot a surface that we're accustomed to like:

$$z = x^2 + y^2$$

parametrically:

```
plot3d([x,y,x^2+y^2],x=-3..3,y=-3..3);
```

Some options to try out:

- `..,y=-3..3,scaling=constrained,style=wireframe);`
- Shading by height is kind of nice: Here, we'll get rid of the lines and use a nice shading:

  `..,y=-3..3,shading=zhue, style=patchnogrid);`

- And if you want the axes, sometimes "boxed" is nice:

  ```
  ..,y=-3..3,shading=zhue, style=patchnogrid,axes=boxed);
  ```

Now try to plot the following surfaces in Maple- Decide on "nice" ranges for u, v:

1. $\langle 2\cos(u), v, 2\sin(u)\rangle$
2. $\langle (2+sin(v))cos(u), (2+sin(v))sin(u), u+\cos(v)\rangle$
3. $\langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\rangle$

Question: What happens if we freeze one of the variables as a constant? What is the resulting graph (a curve or a surface)?

Now we have a mapping from the (u, v) plane to a surface in three dimensions. To put a path (or curve) on the surface, we just have to define a path in the (u, v) plane:

$$u = f(t), \qquad v = g(t)$$

then substitute these into the coordinate mappings. To be explicit, suppose I take $u = \sin(t)$ and $v = cos(\sqrt{3}\,t)$. If I want this path on the surface of the cylinder,

$$\mathbf{r}(t) = \langle 2\cos(\sin(t)), \cos(\sqrt{3}\,t), 2\sin(\sin(t))\rangle$$

Or in Maple:

```
with(plots):
x:=2*cos(u):
y:=v:
z:=2*sin(u):

x1:=subs(u=sin(t),v=cos(sqrt(3)*t),x):
y1:=subs(u=sin(t),v=cos(sqrt(3)*t),y):
z1:=subs(u=sin(t),v=cos(sqrt(3)*t),z):

A:=plot3d([x,y,z],u=-Pi..Pi,v=-1..1,style=wireframe):
B:=spacecurve([x1,y1,z1,t=0..4],thickness=2,color=black);
display3d(A,B);
```

Exercises for Part I

Consider the second surface we defined previously

$$\langle (2+sin(v))cos(u), (2+sin(v))sin(u), u+\cos(v)\rangle$$

where we'll make the domain the rectangle:

$$0 \leq u \leq 4\pi \qquad 0 \leq v \leq 2\pi$$

1. In Maple, plot the surface using the default style or `wireframe` option. Which curves correspond to keeping u constant? Which are keeping v constant?

2. In the rectangular domain, define a line going from $(0,0)$ to $(4\pi, 2\pi)$, then plot this three dimensional curve with the surface (in wire frame).

3. In the rectangular domain, define an interesting curve, then map it to your surface and again plot the result.

Part II: The Torus

We will define a torus (a doughnut-shaped object) by taking a circle in the $xz-$plane,

$$(x+2)^2 + z^2 = 1$$

then we'll rotate this circle through an angle β in the (x, y) plane to create the overall shape.

If we let (x, y, z) be any point on the surface of the torus, then we define α as the angle the point $(x, 0, z)$ makes in the circle $(x+2)^2 + z^2 = 1$, and β is the angle the point $(x, y, 0)$ makes in the $xy-$plane.

We will construct the equations for the torus together.

Questions for Part II

1. What is the natural domain for the torus?

2. Identify where the point $(3, 0, 0)$ and $(-3, 0, 0)$ are in the domain.

3. Identify where the point $(\pi/2, \pi/2)$ should be on the surface of the torus.

4. Find an equation for a path from the point $(0, 0)$ to $(\pi, 2\pi)$, then plot both the path and the torus together.