Surfaces: Lab 3 (Math 235)

Intstructions: For this lab, we'll do all of our work in Maple, and turn in the Maple file with the solutions to the exercises below (with the output removed). This won't be due until the Friday after break, but we will only be having class time for it during the week before break.

No specific partners will be assigned for this lab.

Lines in 2 and 3d

If you want a line that goes from A to B as t goes from 0 to 1, then we parameterize it as:

$$A(1-t)+Bt$$

For example, in three dimensions, to go from $(1,3,2)$ to $(3,4,5)$, then

$$x=(1-t)+3t=2t+1 \qquad y=3(1-t)+4t=t+3 \qquad z=2(1-t)+5t=3t+2$$

Curves in Space

In Calculus III, we saw that space curves are typically defined by a set of parametric equations in time. For example, a helix in three dimensions may be defined by the map F below, or the infamous "pringle" defined by the map G:

$$F(t)=\begin{bmatrix}\cos(t)\\ \sin(t)\\ t\end{bmatrix} \qquad G(t)=\begin{bmatrix}\cos(t)\\ \sin(t)\\ \cos(2t)\end{bmatrix}$$

In any event, the main point is that when the domain is a single variable (time t), the result in two or three dimensions is a *curve*. In the case of a curve in two dimensions, $y=f(x)$, it is easy to parametrize it by taking $x=t$:

$$y=f(x) \quad \Leftrightarrow \quad \begin{bmatrix}t\\ f(t)\end{bmatrix}$$

Plotting curves in space using Maple

The Maple command for plotting curves in 3d is `spacecurve`.

Here is an example, plotting the "pringle" curve:

```
with(plots):
X:=cos(t); Y:=sin(t); Z:=cos(2*t);
spacecurve([X,Y,Z],t=0..3*Pi,color=black,axes=boxed);
```

NOTE for Maple 16: The newer versions of Maple don't react well with our computer lab graphics cards, which can create some problems when rotating the figure.

Arc Length

If we have a curve in two dimensions specified by $y = f(x)$ for $a \leq x \leq b$, then we can determine the arc length using the integral:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

If the curve is in two dimensions and is specified by: $\langle x(t), y(t) \rangle$ for $a \leq t \leq b$, then the arc length is found by computing:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Similarly, if the curve is in three dimensions and is defined by $\langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$, then the arc length is found by:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z(t))^2}\, dt$$

Integration in Maple

Some integrals, especially many arc length integrals, do not have elementary antiderivatives, so the integrals must be numerically approximated. One way to do this in Maple is the following:

```
Digits:=16;
L:=int(exp(x)/x,x=1..7);
```

What you'll get is a special function that Maple has defined-0 The function `Ei` (you can look it up if you're curious about it). To force Maple to give a numerical approximation, it is better to use the *inert form of the integral*, which is:

```
L:=Int(exp(x)/x,x=1..7);  #Inert form of the integral (nothing is computed)
L1:=evalf(L);   #Numerical approximation of L
```

Surfaces

A **surface** in three dimension can be defined directly by

$$z = f(x, y)$$

Just as a curve is a mapping of $\mathbb{R}$ to $\mathbb{R}^2$ or $\mathbb{R}^3$, we can increase the dimension of the domain by 1 and get a surface- That is, a surface can be thought of as a mapping from the plane $\mathbb{R}^2$ to $\mathbb{R}^3$. In this case, we think of x, y and z as functions of *two* variables, for example, u, v.

To see that this generalizes our notion of a surface, consider that if you are given $z = f(x, y)$, then it is easy to parametrize the surface: Let $x = u$ $y = v$, then $z = f(u, v)$. We typically group the three functions in "vector notation":

$$\langle x(u,v), y(u,v), z(u,v)\rangle$$

Example: Convert the surface $z = x^2 + y^2$ into a parametric surface.

SOLUTION: Take $x = u, y = v$ and $z = u^2 + v^2$, so in vector form, we get

$$\langle u, v, u^2 + v^2\rangle$$

Curves on a Surface

We can also put a curve on a surface. Using the previous surface, suppose that $0 \leq u \leq 1$ and $0 \leq v \leq 2$. Now, suppose that u and v are actually functions of one variable, t. Then the surface

$$\langle x(u,v), y(u,v), z(u,v)\rangle$$

becomes a curve in 3-d:

$$\langle x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))\rangle$$

For example, suppose we have the path $u = t$ and $v = 2t$, and we wanted to put the curve on the surface $\langle u, v, u^2 + v^2\rangle$. Then we make the substitution, and the curve is:

$$\langle t, 2t, 5t^2\rangle$$

Here is a sample set of Maple commands that will plot the surface as in "wireframe" (which makes the curve easier to see).

```
with(plots):
S1:=[u, v, u^2+v^2];          #Defines the surface
S2:=subs(u=t,v=2*t,S1);       #Creates the curve on the surface
Plot01:=plot3d(S1,u=-1..1,v=-2..2,style=wireframe);
Plot02:=spacecurve(S2,t=0..1,color=black,thickness=1);
display3d(Plot01,Plot02);
```

Maple Exercises

Solve the following in a Maple Worksheet, and keep it for after Spring Break.

1. Plot the trefoil knot in three dimensions, given parametrically by:

$$x(t) = -10\cos(t) - 2\cos(5t) + 15\sin(2t), \quad y(t) = -15\cos(2t) + 10\sin(t) - 2\sin(5t)$$

$$z(t) = 10\cos(3t)$$

   Then, find its arc length (Hint: You'll need a starting and ending time).

2. (a) Plot the parametric surface in three dimensions:

$$\langle u, \sin(u+v), \sin(v) \rangle \qquad -\pi \leq u \leq \pi, -\pi \leq u \leq \pi$$

   (b) Find a parametric representation for the line in (u, v) coordinates going from $(-3, 2)$ to $(1, 3)$. Put this line on the surface as in the last example.

   (c) Find the arc length of the curve in 3d that you found in the previous part.

3. Plot the surface:

$$\langle \sin(v), \cos(u)\sin(4v), \sin(2u)\sin(4v) \rangle, \quad 0 \leq u \leq 2\pi, -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$$

   - If v is held constant, what kinds of curves do we get in 3d? Plot two of them (with the surface) all in the same graph.
   - If u is held constant, what kinds of curves do we get in 3d? Plot two of them (with the surface) all in the same graph.

4. Given the surface $z = f(x, y)$, suppose we want to approximate the surface close to a point (a, b). Then the tangent plane is given by:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

   And the quadratic approximation is given by:

$$Q(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) +$$
$$\frac{1}{2} f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2} f_{yy}(a, b)(y - b)^2$$

   If $f(x, y) = \mathrm{e}^{-x^2 - y^2}$, and $(a, b) = (0, 0)$, find an expression for the tangent plane L and the quadratic approximation Q, then plot all three surfaces together (the original, the plane and the quadratic approximation).