Surfaces: Lab 3 (Math 235)

Intstructions: For this lab, we'll do all of our work in Maple, and turn in the Maple file with the solutions to the exercises below (with the output removed). This won't be due until the Friday after next week's class, but we won't have class time next week to work on it.

No specific partners will be assigned for this lab.

Lines in 2 and 3d

If you want a line that goes from A to B as t goes from 0 to 1, then we parameterize it as:

$$A(1-t)+Bt$$

where A, B can be points in two, three, or any dimension.

For example, in three dimensions, to go from $(1,3,2)$ to $(3,4,5)$ as time runs from 0 to 1, we would have:

$$(1,3,2)(1-t)+(3,4,5)t, \qquad 0 \le t \le 1$$

is computed coordinate-wise as:

$$x=(1-t)+3t=2t+1 \qquad y=3(1-t)+4t=t+3 \qquad z=2(1-t)+5t=3t+2$$

Now the line is given by $\langle x(t), y(t), z(t)\rangle$.

Curves in Space

In Calculus III, we saw that more generally, space curves are defined by a set of parametric equations in time. For example, a helix in three dimensions may be defined by the map F below, or the infamous "pringle" defined by the map G:

$$F(t)=\begin{bmatrix}\cos(t)\\ \sin(t)\\ t\end{bmatrix} \qquad G(t)=\begin{bmatrix}\cos(t)\\ \sin(t)\\ \cos(2t)\end{bmatrix}$$

(We'll plot these in Maple below).

In any event, the main point is that when the domain is a single variable (time t), the result in two or three dimensions is a *curve*.

In two dimensions, if we're given $y=f(x)$, we can parameterize it easily by taking $x(t)=t$ so that $y(t)=f(t)$.

Plotting curves in space using Maple

We already know how to plot parametrized curves in two dimensions in Maple:

```
plot([x(t), y(t),t=a..b]);
```

so, for example, the following plots the unit circle:

```
plot([cos(t),sin(t),t=0..2*Pi]);
```

The Maple command for plotting curves in 3d is `spacecurve`. Here is an example, plotting the "pringle" curve:

```
with(plots):
X:=cos(t); Y:=sin(t); Z:=cos(2*t);
spacecurve([X,Y,Z],t=0..3*Pi,color=black,axes=boxed);
```

Arc Length

If we have a curve in two dimensions specified by $y = f(x)$ for $a \le x \le b$, then we can determine the arc length using the integral:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

If the curve is in two dimensions and is specified by: $\langle x(t), y(t) \rangle$ for $a \le t \le b$, then the arc length is found by computing:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Similarly, if the curve is in three dimensions and is defined by $\langle x(t), y(t), z(t) \rangle$, $a \le t \le b$, then the arc length is found by:

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z(t))^2}\, dt$$

Integration in Maple

Some integrals, especially many arc length integrals, do not have elementary antiderivatives, so the integrals must be numerically approximated. One way to do this in Maple is the following:

```
Digits:=16;
L:=int(exp(x)/x,x=1..7);
```

What you'll get is a special function that Maple has defined-0 The function `Ei` (you can look it up if you're curious about it). To force Maple to give a numerical approximation, it is better to use the *inert form of the integral*, which is:

```
L:=Int(exp(x)/x,x=1..7);  #Inert form of the integral (nothing is computed)
L1:=evalf(L);   #Numerical approximation of L
```

Surfaces

A **surface** in three dimensions can be defined directly by

$$z = f(x, y)$$

Just as a curve is a mapping of $\mathbb{R}$ to $\mathbb{R}^2$ or $\mathbb{R}^3$, we can increase the dimension of the domain by 1 and get a surface- That is, a surface can be thought of as a mapping from the plane $\mathbb{R}^2$ to $\mathbb{R}^3$. In this case, we think of x, y and z as functions of *two* variables, for example, u, v.

To see that this generalizes our notion of a surface, consider that if you are given $z = f(x, y)$, then it is easy to parametrize the surface: Let $x = u$ $y = v$, then $z = f(u, v)$. We typically group the three functions in "vector notation":

$$\langle x(u, v), y(u, v), z(u, v)\rangle$$

The graph of this is what we call a *surface.*
Example: Convert the surface $z = x^2 + y^2$ into a parametric surface.

SOLUTION: Take $x = u, y = v$ and $z = u^2 + v^2$, so in vector form, we get

$$\langle u, v, u^2 + v^2\rangle$$

Example: Does the graph of the following represent a curve in three dimensions or a surface (in three dimensions)?

- $\langle \cos(t), \sin(t), \cos(2t)\rangle$

  SOLUTION: This is a curve in three dimensions.

- $\langle \cos(u), \sin(v), \cos(uv)\rangle$

  SOLUTION: Because we are mapping (u, v) to three dimensions, this is a surface in 3d.

- $\langle \cos(u), \sin(\pi), \cos(\pi u)\rangle$

  SOLUTION: Because this is mapping u to three dimensions, this is a (space) curve (a curve in 3d).

To plot a parametric surface in Maple, use a slightly different version of the plot command:

```
plot3d([x(u,v), y(u,v), z(u,v)],u=a..b,v=c..d);
```

(Note where the square brackets are).

Curves on a Surface

We can also put a curve on a surface. Using the previous surface, suppose that $0 \leq u \leq 1$ and $0 \leq v \leq 2$. Now, suppose that u and v are actually functions of one variable, t. Then the surface

$$\langle x(u,v), y(u,v), z(u,v)\rangle$$

becomes a curve in 3-d:

$$\langle x(u(t),v(t)), y(u(t),v(t)), z(u(t),v(t))\rangle$$

For example, suppose we have the path $u = t$ and $v = 2t$, and we wanted to put the curve on the surface $\langle u, v, u^2 + v^2\rangle$. Then we make the substitution, and the curve is:

$$\langle t, 2t, 5t^2\rangle$$

Here is a sample set of Maple commands that will plot the surface as in "wireframe" (which makes the curve easier to see).

```
with(plots):
S1:=[u, v, u^2+v^2];         #Defines the surface
S2:=subs(u=t,v=2*t,S1);      #Creates the curve on the surface
Plot01:=plot3d(S1,u=-1..1,v=-2..2,style=wireframe);
Plot02:=spacecurve(S2,t=0..1,color=black,thickness=1);
display3d(Plot01,Plot02);
```

Maple Exercises (Lab 3)

Solve the following in a Maple Worksheet.

1. Plot the trefoil knot in three dimensions, given parametrically by:

   $$x(t) = -10\cos(t) - 2\cos(5t) + 15\sin(2t), \quad y(t) = -15\cos(2t) + 10\sin(t) - 2\sin(5t)$$

   $$z(t) = 10\cos(3t)$$

   The knot is a closed curve, meaning that the curve comes together and starts to repeat itself.

   Find the arc length of the knot. Hint: You'll need a starting and ending time. You might start by guessing from the graph. Double check by evaluating x, y, z at these times (using Maple- Use the `subs` command).

2. (a) Plot the parametric surface in three dimensions:

   $$\langle u, \sin(u+v), \sin(v)\rangle \qquad -\pi \leq u \leq \pi, -\pi \leq u \leq \pi$$

(b) Find a parametric representation for the line in (u, v) coordinates going from $(-3, 2)$ to $(1, 3)$. This will give you two functions, $u(t)$ and $v(t)$. Put this line on the surface in three dimensions as in the last example and plot them together.

(c) Find the arc length of the curve (in 3d) that you found in the previous part.

3. Consider the surface:

$$\langle u\cos(v), u\sin(v), u\rangle \qquad -1 \le u \le 1, \quad 0 \le v \le 2\pi$$

   (a) Plot the surface in 3d.

   (b) If v is held constant, what kinds of curves do we get in 3d? Plot two of them (with the surface) all in the same graph.

   (c) If u is held constant, what kinds of curves do we get in 3d? Plot two of them (with the surface) all in the same graph.

4. Plot the surface:

$$\langle \sin(v), \cos(u)\sin(4v), \sin(2u)\sin(4v)\rangle, \quad 0 \le u \le 2\pi, -\frac{\pi}{2} \le v \le \frac{\pi}{2}$$

   (a) Plot the surface in 3d.

   (b) If v is held constant, what kinds of curves do we get in 3d? Plot two of them (with the surface) all in the same graph.

   (c) If u is held constant, what kinds of curves do we get in 3d? Plot two of them (with the surface) all in the same graph.