Math 235: Calculus Lab

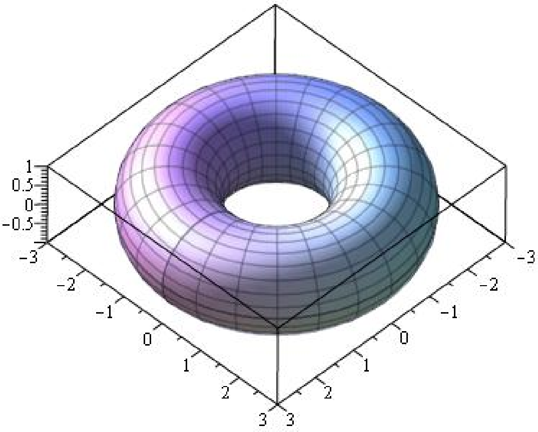
Prof. Doug Hundley

Whitman College

Week 10

The Torus

The torus is an object that looks like a doughnut:

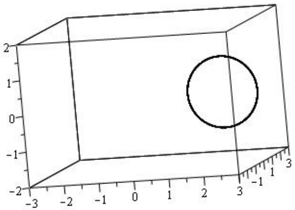


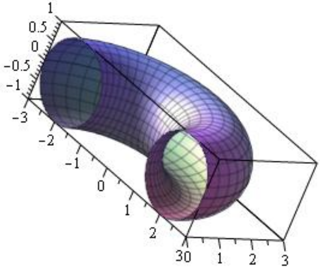
Torus Construction

Our torus is built by taking the graph of the unit circle:

$$(x-2)^2+z^2=1$$

and spinning it around the $z-$axis:





Circles

Any circle with fixed radius K can be parametrized by one number- The central angle, θ.

Circles

Any circle with fixed radius K can be parametrized by one number- The central angle, θ.

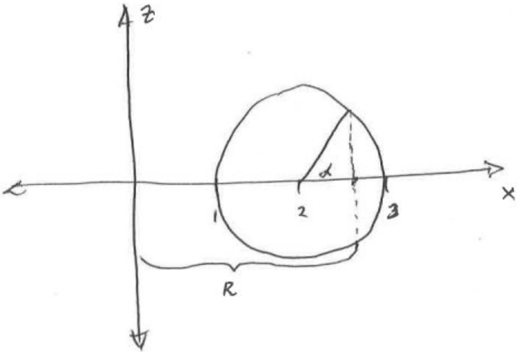
That is, for any point on circle of radius K, we can express that point as:

$$\begin{aligned} x(\theta) &= K\cos(\theta) \\ y(\theta) &= K\sin(\theta) \end{aligned}$$

Obtaining a Point on the Torus

To obtain any point on the surface of the torus we will:

- Start on the circle $(x-2)^2 + z^2 = 1$, and rotate through an angle α.



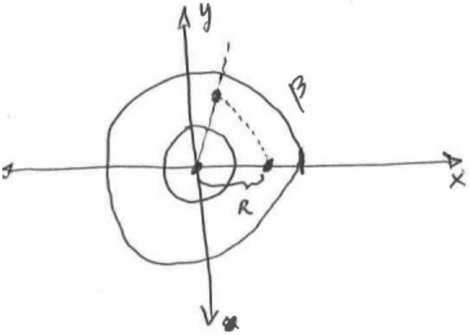
So far, before we rotate into the $xy-$plane,

$$R = \cos(\alpha) + 2$$

$$z = \sin(\alpha)$$

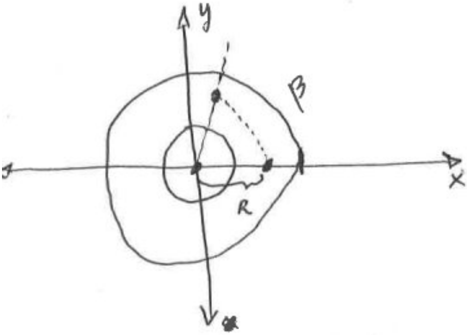
We rotate to get the x and y coordinates...

- In the $xy-$plane, we will then rotate through an angle β.



$$x = R\cos(\beta)$$

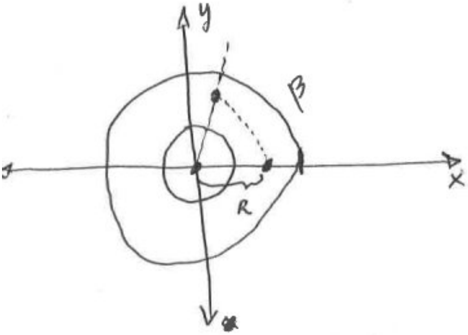
- In the $xy-$plane, we will then rotate through an angle β.



$$x = R\cos(\beta) = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = R\sin(\beta) =$$

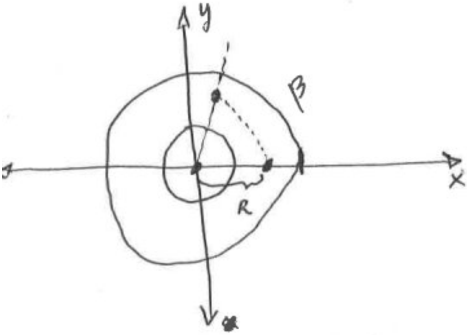
- In the $xy-$plane, we will then rotate through an angle β.



$$x = R\cos(\beta) = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = R\sin(\beta) = \sin(\beta)(\cos(\alpha) + 2)$$

- In the $xy-$plane, we will then rotate through an angle β.



$$x = R\cos(\beta) = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = R\sin(\beta) = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha) \qquad \text{unchanged}$$

Conclusion thus far:

The surface of the torus can be expressed as:

$$x = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha)$$

Example points:

$$\beta = 0, \alpha = 0 \quad \Rightarrow$$

Conclusion thus far:

The surface of the torus can be expressed as:

$$x = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha)$$

Example points:

$$\beta = 0, \alpha = 0 \quad \Rightarrow \quad (3, 0, 0)$$

$$\beta = \pi/2, \alpha = \pi \quad \Rightarrow$$

Conclusion thus far:

The surface of the torus can be expressed as:

$$x = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha)$$

Example points:

$$\beta = 0, \alpha = 0 \quad \Rightarrow \quad (3, 0, 0)$$

$$\beta = \pi/2, \alpha = \pi \quad \Rightarrow \quad (0, 1, 0)$$

$$\beta = \pi, \alpha = 0 \quad \Rightarrow$$

Conclusion thus far:

The surface of the torus can be expressed as:

$$x = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha)$$

Example points:

$$\beta = 0, \alpha = 0 \quad \Rightarrow \quad (3, 0, 0)$$

$$\beta = \pi/2, \alpha = \pi \quad \Rightarrow \quad (0, 1, 0)$$

$$\beta = \pi, \alpha = 0 \quad \Rightarrow \quad (-3, 0, 0)$$

Curves in the (β, α) plane:

If $\beta = \beta(t)$ and $\alpha = \alpha(t)$, then substituting these into

$$x = \cos(\beta)(\cos(\alpha + 2))$$

$$y = \sin(\beta)(\cos(\alpha + 2))$$

$$z = \sin(\alpha)$$

Creates the curve $\langle x(t), y(t), z(t)\rangle$ on the surface.

Path 1

Path 1 keeps $\alpha = 0$ and β ranging from 0 to π. Therefore:

$$\begin{aligned} \beta(t) &= \pi t \\ \alpha(t) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(t) &= 3\cos(\pi t) \\ y(t) &= 3\sin(\pi t) \\ z(t) &= 0 \end{aligned}$$

Path 1

Path 1 keeps $\alpha = 0$ and β ranging from 0 to π. Therefore:

$$\begin{aligned} \beta(t) &= \pi t \\ \alpha(t) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(t) &= 3\cos(\pi t) \\ y(t) &= 3\sin(\pi t) \\ z(t) &= 0 \end{aligned}$$

The path length is

Path 1

Path 1 keeps $\alpha = 0$ and β ranging from 0 to π. Therefore:

$$\begin{aligned} \beta(t) &= \pi t \\ \alpha(t) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(t) &= 3\cos(\pi t) \\ y(t) &= 3\sin(\pi t) \\ z(t) &= 0 \end{aligned}$$

The path length is half the circumference of a circle of radius 3:

Path 1

Path 1 keeps $\alpha = 0$ and β ranging from 0 to π. Therefore:

$$\begin{aligned} \beta(t) &= \pi t \\ \alpha(t) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(t) &= 3\cos(\pi t) \\ y(t) &= 3\sin(\pi t) \\ z(t) &= 0 \end{aligned}$$

The path length is half the circumference of a circle of radius 3:

$$3\pi$$

.

[Plot in the (β, α) plane]

Path 2

Path 2 is actually 3 paths:

$$(0,0) \to (0,\pi) \to (\pi,\pi) \to (\pi,0)$$

Path 2A: $\beta = 0$, $\alpha = \pi t$
Path 2B: $\beta = \pi t$, $\alpha = \pi$
Path 2C: $\beta = \pi$, $\alpha = \pi(1-t)$

In Maple, do these separately, and plot them all together.

Path Length: $\pi + \pi + \pi = 3\pi$

[plot in (β, α) plane]

Path 3

In this case, take β from 0 to π. Then α will go from 0 to 2π.

$$\begin{aligned} \beta(t) &= \pi t \\ \alpha(t) &= 2\pi t \end{aligned} \quad \Rightarrow \quad \begin{aligned} x(t) &= \cos(\pi t)(\cos(2\pi t) + 2) \\ y(t) &= \sin(\pi t)(\cos(2\pi t) + 2) \\ z(t) &= \sin(2\pi t) \end{aligned}$$