Math 235: Calculus Lab

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Week 11

This Week:

- A piecewise defined path.
- Optimizing over a family of paths.
- Discussion of the Lab.

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Next Week: Nothing new. Continue (or start) typesetting your paper. Be sure to think about the comments from earlier papers!

A Piecewise Defined Path

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- $(0, 0)$ to $(0, \pi)$, Parametrization:

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- $(0,\pi)$ to (π,π), Parametrization: $\beta(t) = \pi t$, $\alpha(t) = \pi$
- (π, pi) to $(\pi, 0)$, Parametrization:

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- $(0,\pi)$ to (π,π), Parametrization: $\beta(t) = \pi t$, $\alpha(t) = \pi$
- (π, pi) to $(\pi, 0)$, Parametrization: $\beta(t) = \pi$, $\alpha(t) = \pi(1-t)$

Implementation in Maple (See worksheet)

The Torus

```
with(plots):
f:=cos(beta)*(cos(alpha)+2):
g:=sin(beta)*(cos(alpha)+2):
h:=sin(alpha):
Torus1:=plot3d([f,g,h],alpha=0..2*Pi,beta=0..2*Pi,
    scaling=constrained,style=wireframe,shading=ZHUE):
```

The Paths

```
xt:=0: yt:=Pi*t:
Path2AF:=subs({beta=xt,alpha=yt},f),
         subs({beta=xt,alpha=yt},g),
         subs({beta=xt, alpha=yt},h):
xt:=Pi*t:  yt:=Pi:
Path2BF:=subs({beta=xt,alpha=yt},f),
         subs({beta=xt,alpha=yt},g),
         subs({beta=xt, alpha=yt},h):
xt:=Pi:  yt:=Pi*t:
Path2CF:=subs({beta=xt,alpha=yt},f),
         subs({beta=xt,alpha=yt},g),
         subs({beta=xt, alpha=yt},h):
```

The Path Lengths

```
dP1:=diff([Path2AF],t);
dP2:=diff([Path2BF],t);
dP3:=diff([Path2CF], t);
Integrand1:=simplify(dP1[1]^2+dP1[2]^2+dP1[3]^2);
Integrand2:=simplify(dP2[1]^2+dP2[2]^2+dP2[3]^2);
Integrand3:=simplify(dP3[1]^2+dP3[2]^2+dP3[3]^2);
PathLength:=int(sqrt(Integrand1),t=0..1)+
            int(sqrt(Integrand2),t=0..1)+
            int(sqrt(Integrand3),t=0..1);
```

Why the square brackets?

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            int(sqrt(Integrand3),t=0..1);
```

Why the square brackets?
The variable `Path2AF` is a list of three things (with no delimiters around them).
Putting square brackets around it make it one "thing", so the derivative operation will work.

Graphics

```
Path2A:=spacecurve([Path2AF,t=0..1],color=black,
        thickness=5):
Path2B:=spacecurve([Path2BF,t=0..1],color=black,
        thickness=5):
Path2C:=spacecurve([Path2CF,t=0..1],color=black,
        thickness=5):
display3d({Torus1, Path2A, Path2B, Path2C});
```

The main idea for the next path:
Rather than going from $(0, 0)$ to $(\pi/2, \pi)$ as we did in Path 3, let β go from 0 to an unknown value, k as α runs from 0 to π.

Path 4 in the (β, α) plane:

$$(0, 0) \to (k, \pi) \to (?, \pi) \to (\pi, 0)$$

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Path 4A: $(0,0)$ to (k,π)

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Path 4A: $(0, 0)$ to (k, π)
Path 4B: (k, π) to $(\pi - k, \pi)$

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Path 4B: (k,π) to $(\pi - k,\pi)$
Path 4C: $(\pi - k,\pi)$ to $(\pi,0)$.

Make the appropriate changes to the Maple file. What values should we allow k to take?

Once we get the paths:

- Path 4A: `xt:=k*t;` `yt:=Pi*t`
- Path 4B: `xt:= k*(1-t)+(Pi-k)*t`
- Path 4C: `xt:=(Pi-k)*(1-t)+Pi*t` `yt:=Pi*(1-t)`

Be Sure To Use capital I for the Integral!

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Be Sure To Use capital I for the Integral!

- The path length depends on k. Plot it!
- Now find the optimal value of the path length!

Sample solution in Maple:

```
xt:=k*t; yt:=Pi*t;
Path3AF:=subs...  (Same as before)
xt:=k*(1-t)+(Pi-k)*t;  yt:=Pi;
Path3BF:=subs...  (Same as before)
xt:=(Pi-k)*(1-t)+Pi*t; yt:=Pi*(1-t);
Path3CF:=subs...  (Same as before)
dP1:=diff([Path3AF],t); dP2:=diff([Path3BF],t); dP3:=diff(
```

```
Integrand1:=simplify(dP1[1]^2+dP1[2]^2+dP1[3]^2);
Integrand2:=simplify(dP2[1]^2+dP2[2]^2+dP2[3]^2);
Integrand3:=simplify(dP3[1]^2+dP3[2]^2+dP3[3]^2);
PathLength:=Int(sqrt(Integrand1),t=0..1)+
            Int(sqrt(Integrand2),t=0..1)+
            Int(sqrt(Integrand3),t=0..1);
plot(PathLength,k=0..Pi/2);
```

What should your graph be? At $k = 0$? $k = \pi/2$?

To find the minimum, we set the derivative to zero. We'll need to do it numerically, so we need an approximate answer.

- Find an expression for the derivative.
- Plot the derivative to get an approximate answer.
- Use the approximation in `fsolve`
- Find the numerical value of the best path.

In Maple:

```
dPath:=diff(PathLength,k);
plot(dPath,k=0..Pi/2);
BestK:=fsolve(dPath=0,k=Pi/4..5*Pi/16);
evalf(subs(k=BestK,PathLength));
```

Continuing: Plot the resulting path in Maple

```
Path3A:=subs(k=BestK,[Path3AF]);
Path3B:=subs(k=BestK,[Path3BF]);
Path3C:=subs(k=BestK,[Path3CF]);
P1:=spacecurve(Path3A,t=0..1,color=black,thickness=5):
P2:=spacecurve(Path3B,t=0..1,color=black,thickness=5):
P3:=spacecurve(Path3C,t=0..1,color=black,thickness=5):
display3d(Torus1,P1,P2,P3);
```