

# Math 235: Calculus Lab

Prof. Doug Hundley

Whitman College

Week 9

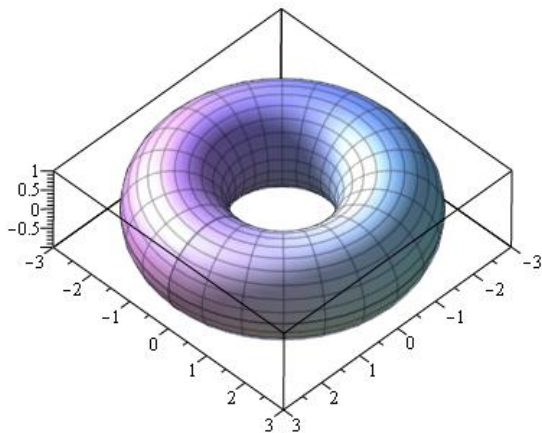
Ants and Three Paths: The goal is to go from  $(3, 0, 0)$  to  $(-3, 0, 0)$  on a path with the shortest distance.

Trial Runs:

1. Go along the “equator” of the doughnut.
2. Go directly to the inside circle, then go around, then climb back out.
3. Don't go directly to the inside circle- Instead, go around both circles simultaneously.

# The Torus

The torus is an object that looks like a doughnut:

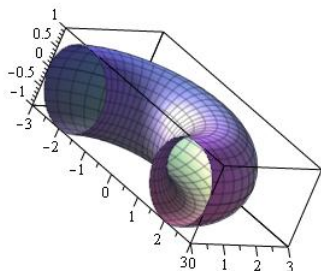
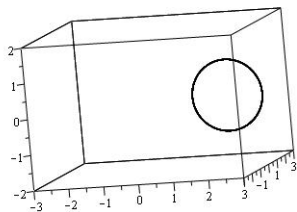


# Torus Construction

Our torus is built by taking the graph of the unit circle:

$$(x - 2)^2 + z^2 = 1$$

and spinning it around the  $z$ -axis:



# Circles

Any circle with fixed radius  $K$  can be parametrized by one number-  
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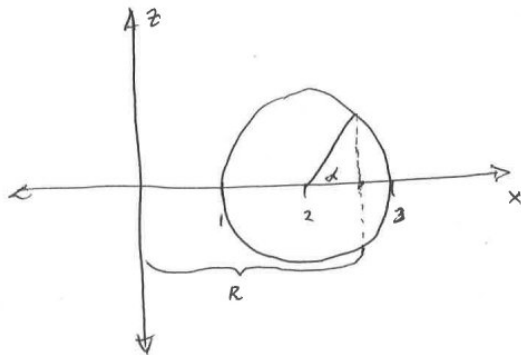
That is, for any point on circle of radius  $K$ , we can express that point as:

$$\begin{aligned}x(\theta) &= K \cos(\theta) \\y(\theta) &= K \sin(\theta)\end{aligned}$$

# Obtaining a Point on the Torus

To obtain any point on the surface of the torus we will:

- Start on the circle  $(x - 2)^2 + z^2 = 1$ , and rotate through an angle  $\alpha$ .



So far, before we rotate into the  $xy$ -plane,

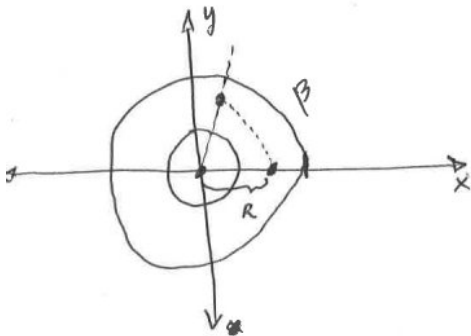
$$R = \cos(\alpha) + 2$$

$$z = \sin(\alpha)$$

We rotate to get the  $x$  and  $y$  coordinates...

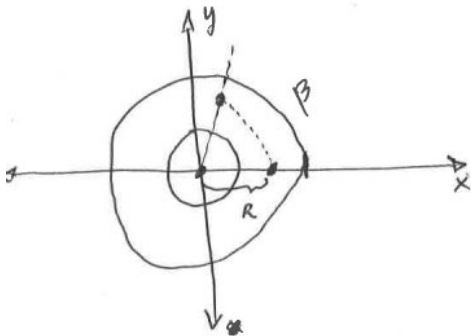


- In the  $xy$ -plane, we will then rotate through an angle  $\beta$ .



$$x = R \cos(\beta)$$

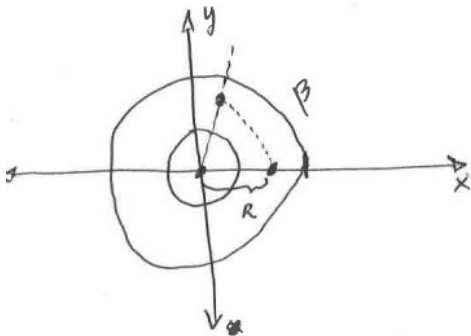
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$$x = R \cos(\beta) = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = R \sin(\beta) =$$

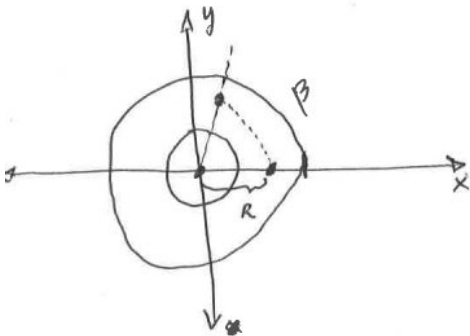
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$$z = \sin(\alpha) \quad \text{unchanged}$$

## Conclusion thus far:

The surface of the torus can be expressed as:

$$x = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha)$$

Example points:

$$\beta = 0, \alpha = 0 \quad \Rightarrow$$

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$$\beta = \pi, \alpha = 0 \quad \Rightarrow \quad (-3, 0, 0)$$



Curves in the  $(\beta, \alpha)$  plane:

If  $\beta = \beta(t)$  and  $\alpha = \alpha(t)$ , then substituting these into

$$x = \cos(\beta)(\cos(\alpha) + 2)$$

$$y = \sin(\beta)(\cos(\alpha) + 2)$$

$$z = \sin(\alpha)$$

Creates the curve  $\langle x(t), y(t), z(t) \rangle$  on the surface.

## Path 1

Path 1 keeps  $\alpha = 0$  and  $\beta$  ranging from 0 to  $\pi$ . Therefore:

$$\begin{array}{ll} \beta(t) = \pi t & \\ \alpha(t) = 0 & \Rightarrow \end{array} \quad \begin{array}{ll} x(t) &= 3 \cos(\pi t) \\ y(t) &= 3 \sin(\pi t) \\ z(t) &= 0 \end{array}$$

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The path length is half the circumference of a circle of radius 3:

$$3\pi$$

.

[Plot in the  $(\beta, \alpha)$  plane]

## Path 2

Path 2 is actually 3 paths:

$$(0, 0) \rightarrow (0, \pi) \rightarrow (\pi, \pi) \rightarrow (\pi, 0)$$

Path 2A:  $\beta = 0, \alpha = \pi t$

Path 2B:  $\beta = \pi t, \alpha = \pi$

Path 2C:  $\beta = \pi, \alpha = \pi(1 - t)$

In Maple, do these separately, and plot them all together.

Path Length:  $\pi + \pi + \pi = 3\pi$

[plot in  $(\beta, \alpha)$  plane]

## Path 3

In this case, take  $\beta$  from 0 to  $\pi$ . Then  $\alpha$  will go from 0 to  $2\pi$ .

$$\begin{array}{ll} \beta(t) = \pi t & \\ \alpha(t) = 2\pi t & \Rightarrow \end{array} \quad \begin{array}{l} x(t) = \cos(\pi t)(\cos(2\pi t) + 2) \\ y(t) = \sin(\pi t)(\cos(2\pi t) + 2) \\ z(t) = \sin(2\pi t) \end{array}$$