Prelab: The Action of a Matrix

Recall that a system of equations can be expressed as a matrix-vector product:

$$\begin{aligned} ax_1 + bx_2 + e &= y_1 \\ cx_1 + dy_1 + f &= y_2 \end{aligned} \Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Leftrightarrow A\mathbf{x} + \mathbf{b} = \mathbf{y}$$

We will call A a matrix and $\mathbf{x}, \mathbf{y}$ vectors.
Notice that if we think of $A, \mathbf{b}$ as being fixed, we can "input" a vector $\mathbf{x}$ and "output" a vector $\mathbf{y}$. For example, let

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

then

$$A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1)(-3) + (3)(2) + 0 \\ (-1)(-3) + (2)(0) + 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

in which case the ordered pair $(-3, 2)$ was input, and $(6, 3)$ was output. In this pre-lab, we want to investigate the 2×2 matrices as functions from the plane (the domain) to the plane (the range),

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{b} = \mathbf{y}$$

We cannot graph these functions directly, since that would require four dimensions (two for the domain, two for the range). Instead, we can consider the action of the function on the points of a unit square. In Matlab, we will construct a set of points and map the points using the matrix- for example, type the following:

```
X=rand(2,1000);
A=[1,3 ; -1, 0];
b=[-1;2];

for j=1:1000
Y(:,j)=A*X(:,j)+b;
end

plot(Y(1,:),Y(2,:),'r.'); %A plot of the action of A on the data
hold off
```

The blue dots correspond to domain points, and the red dots correspond to range (or image) points.

1 The Action of a Matrix

- Scaling: A horizontal (left-right) scaling by α, vertical scaling (up-down) by β:

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

- Rotation CCW through angle θ: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
- Flip across the line $y = x$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- Reflect across the $x-$axis ($y \to -y$): $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

2 Matrix Composition/Multiplication

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let us apply matrix A to the vector $\mathbf{x}$ first, then apply the matrix B:

$$A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

Now apply B to the result (verify this!):

$$B(A\mathbf{x}) = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} (ae + cf)x_1 + (be + df)x_2 \\ (ag + ch)x_1 + (bg + dh)x_2 \end{bmatrix}$$

which is equal to:

$$\begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix} \mathbf{x}$$

We define this matrix to be the matrix product BA:

$$BA = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix}$$

Note that this could be written as:

$$BA = \left[B \begin{bmatrix} a \\ c \end{bmatrix} \quad B \begin{bmatrix} b \\ d \end{bmatrix} \right]$$

SUMMARY: Matrix multiplication is function composition. That is, the matrix product BA represents A being applied to $\mathbf{x}$ first, then B.
In Matlab, if A, B are 2×2 matrices, the matrix product is `B*A`.

3 Example

Find the affine function that performs the following actions (in order): Shift down 2, to forward 3. Next scale vertically by 1/3, horizontally by 1/2. Finally, rotate the result by 30 degrees CCW. Finally, translate up by 2, backwards 3 units.

- Let $\mathbf{x}$ be a point in $\mathbb{R}^2$.
- Shift:

$$\mathbf{x} - \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

- Scale:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \left(\mathbf{x} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right)$$

- Rotate and shift:

$$\begin{bmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{bmatrix} \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \left(\mathbf{x} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right) \right)$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \left(\mathbf{x} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right) \right) + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

which we can think of as:

$$B(A(\mathbf{x} - \mathbf{b})) + \mathbf{c}$$

$$(BA)\mathbf{x} + (-BA\mathbf{b} + \mathbf{c})$$

In Matlab,

```
b=[3;-2];
A=[1/2,0;0,3];
B=[cos(pi/6),-sin(pi/6);sin(pi/6),cos(pi/6)];
c=[-3;2];
```

so that the affine function $\hat{A}\mathbf{x} + \hat{b}$ is:

```
hatA=B*A;
hatb= - B*A*b + c;
```

4 Self Similarity

An object is said to be *self-similar* if it is composed of smaller versions of itself. We saw some examples in the movie- Cauliflower and Broccoli, trees and clouds are all self-similar in some degree.
We can build self similar objects by using affine transformations. Think of each affine transformation as creating a smaller copy of the whole. For example, the Serpinski Gasket (the object we created in the last lab) is self similar by using 3 smaller copies of itself.

5 The Lab

Find the affine transformations that will create the self similar objects shown. Additionally, create a self similar word (make it short) by using some graph paper to find the affine maps. We'll go through an example in class.
For the final lab, turn in the resulting pictures and scripts used to create them.