Calculus Lab II, Spring 2004
Pre-Lab 1: Review of Power Series

Next week I'll give you a short lab based on power series. This week, I would like for you to review the basics on power series and Taylor series. Include your typewritten (or graphical) answers to the following questions in the lab writeup.

1. A *power series based at* $x = a$ is a function of the form:

$$P(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

   where c_n is a constant, for all n. What does it mean to take an "infinite" sum?

2. What is the radius of convergence for a power series? What is the "Ratio Test"?

3. Can you differentiate/integrate a power series? What is the result after doing this to $P(x)$ defined in Problem 1?

4. What is a Taylor series? How is it different than a Maclaurin series?

5. What is the meaning of "n^{th} Taylor polynomial"? What is the meaning of $R_n(x)$ (used in Stewart's Calc text)?

6. How does using a Taylor series allow us to integrate functions we otherwise could not?

7. Do *all* functions have a power series representation? How can you tell if it does?

8. It is shown in Calculus that, if $|x| < 1$, then

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (1)$$

   Show graphically that the partial sums approximate $1/(1-x)$ by plotting some. That is, define:

$$f_0(x) = \frac{1}{1-x}, \quad f_3(x) = 1 + x + x^2 + x^3,$$

$$f_6(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6, \text{ etc.}$$

   and plot f_0 along with functions of increasing degree. Careful in defining the domain and range windows!

9. This refers to the previous problem: Integrate both sides of Equation 1 and repeat the graphs.

Power Series and Maple

In the example below, we use Maple and the Taylor series to approximate the antiderivative of e^{x^3}

```
F:=int(exp(x^3),x);
```

Your answer has $\Gamma(x)$ in it- this is the Gamma function, which is itself defined as an integral. To satisfy your curiousity, it is defined as:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

If n is a positive integer, then

$$\Gamma(n) = (n-1)!$$

The Γ function is therefore a generalization of the factorial for non-integers (and it is used extensively in statistics).

Getting back to the problem, we'd like an approximation to the antiderivative of e^{x^3}. Given Maple's output, we continue:

```
Ft:=taylor(F,x=0,12);
     x+1/4*x^4+1/14*x^7+1/60*x^10+O(x^13)
```

Maple uses the "Big-Oh" notation to tell us that the next term up will be x^{13}. There is a formal definition for the "Big-Oh", but an intuitive idea is sufficient for now (we define it formally in Numerical Analysis). We now need to convert this power series into a regular polynomial so that we can plot it, differentiate it, integrate it, etc. This is done as:

```
Fp:=convert(Ft,polynom);
```

Is this a very good approximation? One way to check is to compare the derivative of this to the series for e^{x^3}. That is, compare the Maple output for:

```
diff(Fp,x);
taylor(exp(x^3),x,11);
```