Lab 1: Taylor Approximations to Functions

1 Introduction to the Lab

In the pre-lab, we discussed the Taylor expansion for a function at a point, $x = a$ as:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

We can view this as a decomposition of $f(x)$ into pieces, or *building blocks.* That is, the building blocks are a constant, a line, a parabola, a cubic, etc.

In the lab, we will:

1. Investigate a special function which is NOT representable by its Taylor series. This will also show what needs to happen in order for a function to be representable by its power series.

2. Introduce multivariate Taylor Series, and compare the multivariate polynomial to the original graph.

Here we go!

2 The Lab

1. Consider the function:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Compute the derivative of f if $x \neq 0$, and at $x = 0$. Note: Be careful at $x = 0$! You can use Maple to compute any limits you need.

   (b) Compute the second derivative of f, again for the two cases.

   (c) Compute the third derivative of f, again for the two cases.

   (d) Based on these computations, what do you think the n^{th} derivative of f at zero will be?

   (e) Plot (on the same coordinate axes) the third, fourth and fifth derivatives of f. What do you notice happens as you take more and more derivatives?

   (f) Now we get to the big question: Is f representable by its power series at $x = 0$? Why not? HINT: There is a formula to estimate the remainder of Taylor's polynomial:

Given a function f with n continuous derivatives on the interval $[a, b]$ and its $(n + 1)$st derivative defined on (a, b), Taylor's formula with remainder is:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\beta_n)}{(n+1)!}(x-a)^{n+1}$$

Where β_n is some point in (a, b).

What is it about the derivatives of f that tells us that its associated Taylor polynomial will not converge to f (except at the trivial point, $x = 0$)?

2. Let's look at the multivariate version of Taylor's polynomial. For visualization purposes, we'll only look at functions $z = f(x, y)$.

   The Taylor polynomial for $z = f(x, y)$ at (a, b) is given below. The function and all derivatives are evaluated at the point (a, b):

$$f + f_x(x-a) + f_y(y-b) +$$

$$\frac{1}{2}\left(f_{xx}(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2\right) +$$

$$\frac{1}{3!}\left(f_{xxx}(x-a)^3 + 3f_{xxy}(x-a)^2(y-b) + 3f_{xyy}(x-a)(y-b)^2 + f_{yyy}(y-b)^3\right) + \ldots$$

   Note that the coefficients for the partials forms Pascal's Triangle.

   (a) Why do you think there is a two in front of f_{xy}, and 3 in front of f_{xxy} and f_{xyy}? What would be the coefficients in front of $f_{xxxy}, f_{xxyy}, f_{xyyy}$?

   (b) You can get Maple to compute the multivariate Taylor approximation using `mtaylor`. Use Maple's `help` feature to determine how to use this command (in particular, try out the examples!).

   (c) Consider the "peaks" function, defined by:

$$z = 3(1-x)^2 \mathrm{e}^{-x^2-(y+1)^2} - 10\left(\frac{1}{5}x - x^3 - y^5\right)\mathrm{e}^{-x^2-y^2} - \frac{1}{3}\mathrm{e}^{-(x+1)^2-y^2}$$

   Graph the function and:

   - At the origin, also plot higher and higher degree Taylor approximations on the window: $-1 \le x \le 1$, $-1 \le y \le 1$.
   - At $(1, 0)$, plot z together with the 4th degree Taylor polynomial. You choose the window size so that the Taylor approximation is "good".
   - At $(0, -2)$, plot z together with the 4th degree Taylor polynomial. You choose the window size so that the Taylor approximation is "good".

   (NOTE: To plot in 3-d, use `plot3d`. You'll need to recall how to do multiple plots at once, or overlay plots using `display3d`).