HOW TO TUNE A RADIO

PUT YOUR NAME HERE

1. Introduction

In the previous lab, we looked in some detail as to when and how functions could be expressed in a Taylor series. The big idea was that very general functions could be built from summing polynomials together (x, x^2, x^3, etc).

Now we look at another way of building functions- from sines and cosines- such sums are called Fourier Series. We'll look at this in the context of understanding how a radio tunes in a signal[1]. To stay closer to our central theme of Fourier Series, we'll focus on *amplitude modulation* (or AM) rather than *frequency modulation* (or FM).

To tune a radio, we turn the dial to a specific frequency. Somehow, that picks out the signal from a specific radio station, decodes it, and we can hear music or news.

Every radio station produces a signal, $f(t)$. We will not go into the physics of what that signal is; we're just taking it as some function of t. The antenna picks up a sum of signals from lots of radio stations; suppose there are n of them. Then the signal picked up is the sum:

$$F(t) = f_1(t) + f_2(t) + \ldots + f_n(t)$$

where $f_j(t)$ is the signal produced by radio station j. Let's be even more specific: Suppose I want to listen to $f_1(t)$. The problem is that I'm picking up a sum of signals- How do I isolate my music in f_1?

- **QUESTION 0:** If we know only $f(t)$, where $f(t) = g(t) + h(t)$, can we recover $g(t)$?

In this lab, we'll see that radio signals are NOT general sums, but are summed in a special way that allows us to isolate desired signals while deleting undesired signals. Moreover, we will be introduced to an amazing discovery of Joseph Fourier, made in about 1807- that most functions can be written in terms of sines and cosines.

2. Trigonometric Identities and Integrals

To get ready for what we're about to do, use a calc book (or look on the web), and answer the following questions:

(1) Let $n \neq 0$. What is $\int_0^{\pi} \cos(nt)\, dt$ (do this by hand). Plot $\cos(nt)$ for $n = 1, 2, 3$. How does changing n change $\cos(nt)$?

[1] This project is taken from "How to Tune a Radio", Applications in Calculus, MAA Notes Number 29, 1997

(2) Look up the following identities (either in a calc book or on the web under "product to sum formulas"):
- $\sin(A)\sin(B) =$
- $\cos(A)\cos(B) =$

(3) Compute the following, using your previous identities. You can use Maple to double check your answer, but also provide all of the details by hand.
- $\int_0^\pi \sin(nt)\sin(mt)\,dt$, where $n \neq m$ and where $n = m$.
- $\int_0^\pi \cos(nt)\cos(mt)\,dt$, where $n \neq m$ and where $n = m$.

(4) Plot $t\sin(20t)$, $t^2\sin(20t)$, $\sin(t)\sin(20t)$. Can you find the functions t, t^2, and $\sin(t)$ in the graphs? (Explain).

NOTE: Functions of the form $g(t)\sin(nt)$ where g is slowly varying with respect to the rapidly varying $\sin(nt)$ are called Amplitude Modulated signals (or AM signals)

(5) Let a_k be some real constant.
- Rewrite the expression
$$a_k \sin(kt)\sin(At)$$
using the trig identities found previously.
- Using your previous answer as a substitution, determine the values of m for which the following integral will NOT be zero (and in those cases, say what the integral will be):
$$a_k \int_0^\pi \sin(kt)\sin(At)\cos(mt)\,dt$$

3. Radio Signals: A Specific Example

Suppose we have three radio stations, R_1, R_2, R_3. Each of them are assigned a rapidly varying carrier wave, $\sin(nt)$. For this example, let's take $\sin(8t), \sin(16t), \sin(24t)$, respectively. The three radio stations want to transmit their programming, and let's call these S_1, S_2 and S_3, where

$$S_1(t) = 3\sin(t) - 2\sin(t)$$

$$S_2(t) = 5\sin(t) + 6\sin(2t)$$

$$S_3(t) = 4\sin(t) + 7\sin(2t)$$

The radio stations actually send $S_1(t)\sin(8t)$, $S_2(t)\sin(16t)$ and $S_3(t)\sin(24t)$. Our radio antenna picks up the sum,

$$f(t) = S_1(t)\sin(8t) + S_2(t)\sin(16t) + S_3(t)\sin(24t)$$

Let's suppose we want to pick out signal S_2. We know that the incoming signal is of the form $a_1\sin(t) + a_2\sin(2t)$, so we only need a_1, a_2 in order to put together the signal from the second radio station.

Exercise 6: Find a constant k and a value for m so that:

$$k\int_0^\pi f(t)\cos(mt)\,dt = 5 \quad (a_1 \text{ for radio station 2})$$

and k and a value for m so that:

$$k\int_0^\pi f(t)\cos(mt)\,dt = 6 \quad (a_2 \text{ for radio station 2})$$

HINT: Use the second part of your answer to question (5) above!

Congratulations! You have recovered the signal from the second radio station! That's AM radio.

Exercise 7: Repeat Exercise 6 to recover the signal from the third radio station.

4. The Big Picture

The procedure we described works pretty well, as long as:

- All music, speech, etc. can be written as:

$$a_1\sin(t) + a_2\sin(2t) + \ldots a_k\sin(kt)$$

  for some value of k.
- The carrier signal has a frequency that is much larger than the k in the previous item (actually, about twice as big).
- The space between two carrier waves is large enough so that we have room to compute our m's (like in Exercise 6 and 7).
- There exists electronics components that can perform the desired integrations (they do exist, this is how a radio tunes a signal).

We won't discuss the last three items (you might ask a physicist!). Rather, we'll focus on the first item.

There is a beautiful theorem by Fourier, where he said that any continuous function can be written as a sum of sines and cosines. We'll keep our discussion to the Fourier sine series (versus the complete Fourier series).

I want to write a function $g(t)$ defined on $(0,\pi)$ as a sum of sine functions. Therefore,

$$g(t) = a_1\sin(t) + a_2\sin(2t) + a_3\sin(3t) + \ldots$$

What should a_k be for $k = 1, 2, \ldots$? We can compute it by observing that integrating both sides of our equation gives:

$$\int_0^\pi g(t)\sin(kt)\,dt = 0 + 0 + \ldots + a_k \cdot \frac{\pi}{2} + 0 + 0 + \ldots$$

so that

$$a_k = \frac{2}{\pi}\int_0^\pi g(t)\sin(kt)\,dt$$

Now we see what radio stations do- they transmit the sound by transmitting a truncated Fourier sine series (you would keep enough terms of the sum to get a good sound).

Let's see how this works by downloading the Fourier sine series worksheet on our class website. Read over the worksheet and see if you can understand what the new Maple commands are doing!

Exercise: Change the worksheet to see the Fourier sine series for a different $g(t)$- Be creative!