Lab 1: Analysis of ODEs, Part I

1. Purpose

The purpose of this lab is for us to see what a differential equation is. To be more specific, what does the equation

$$y' = f(t, y)$$

mean, and what conclusions may we draw from it? In particular, if we make some simplifications to two special cases:

$$y' = f(t) \text{ or } y' = f(y)$$

what extra information might we get?

We will answer these questions by looking at the *direction field* that is plotted via Maple.

2. Define a Differential Equation and Direction Fields

To define a differential equation in Maple, we'll assign it to a variable for easy manipulation later. The derivative can take one of two forms, depending on if the input is an expression or a function.

> **EXAMPLE 1:** (See page 41 of Boyce and Diprima) Plot the direction field for
> $$y' = \frac{x^2}{1 - y^2}$$

First, define the differential equation and assign it to a variable (in this case, `diffeqn`). In the first example, we treat $y(x)$ as an expression in x:

```
diffeqn:=diff(y(x),x)=x^2/(1-(y(x))^2);
```

Alternatively, you can define it in terms of functions. In this case, we use the uppercase D form of the derivative:

```
diffeqn2:=D(y)(x)=x^2/(1-(y(x))^2);
```

And now we can plot the direction field as either:

```
with(DEtools):
DEplot(diffeqn,y(x),x=-4..4,y=-4..4);
DEplot(diffeqn2,y(x),x=-4..4,y=-4..4);
```

You'll notice that we needed to tell Maple (1) the differential equation, (2) What the independent and dependent variables are (the statement $y(x)$ does this), and (3) the window size that we wanted to plot.

We'll now take a look at some optional commands. Try redoing the last line, and make a note at how the plot changes:

```
DEplot(diffeqn,y(x),x=-4..4,y=-4..4, title='First Plot',arrows=LARGE);

DEplot(diffeqn,y(x),x=-4..4,y=-4..4, dirgrid=[30,30]);
```

We can use the same command to plot some sample solution curves along with the direction field. To do this, we will have to include some initial values.

EXAMPLE 2: Plot the direction field for the given differential equation, together with the three sample solutions through $(1,2),(1,0)$ and $(1,1)$, if

$$ty' + 2y = 4t^2$$

First, define the differential equation, then plot:

```
diffeqn:=t*diff(y(t),t)+2*y(t)=4t^2;
DEplot(diffeqn,y(t),t=0.01..2,[[y(1)=2],[y(1)=0],[y(1)=1]],y=-2..4);
```

QUESTION: What happens if $t = 0$?

3. Lab Questions

(1) Plot the following direction fields:
  (a) The form for these is: $y' = f(t,y)$
    (i) $y' = 1 + 3\sin(t) + y$
    (ii) $y' = \frac{1}{2}y + 2\cos(t)$
  (b) The form for these is: $y' = f(y)$. These equations are known as *autonomous equations*, since the differential equation doesn't depend explicitly on time.
    (i) $y' = y(3-y)$
    (ii) $y' = y(y-1)(y-2)$
  (c) The form for these is: $y' = f(t)$
    (i) $y' = 3t^2 - 2$
    (ii) $y' = \frac{\sin(t)}{t}$

  Analyze your output from this section. Given a direction field, is it possible to determine the form of the differential equation (that is, is it $y' = f(y)$, $y' = f(t)$, or something else)?

(2) In Problem 1(b), is it possible to determine the behavior of the solution, y, as $t \to \infty$? Describe this behavior in terms of its dependence on the initial value, $y(0)$.

(3) Let

$$y' = 3\sin(t) + 1 + y$$

  Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

  Think about this question and I'll give you a new command next week!