SYSTEMS OF DIFFERENTIAL EQUATIONS

1. A Case Study: Competing Species

Let $x(t)$ and $y(t)$ be two species that compete for the same resources. Consider the following model of the rate of change of the populations:

$$\begin{aligned} \dot{x} &= 2x\left(1 - \tfrac{1}{2}x\right) - xy \\ \dot{y} &= 3y\left(1 - \tfrac{1}{3}y\right) - 2xy \end{aligned}$$

Before going any further, we want to make some observations:

- This is a system of first order equations. They are *coupled* in the sense that $\dot{x}$ depends on y, and $\dot{y}$ depends on x.
- This is a nonlinear system of equations.
- The derivatives do not depend explicitly on time. Therefore this is a system of autonomous differential equations.
- Because the system is autonomous, the behavior of solutions is organized about the equilibria (remember that we discussed this in the case that $y' = f(y)$) . In this case, to find the equilibria, we must set $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously and solve for values of x, y. We will see how to do this in Maple.
- The *phase plot* will be a plot of y versus x. We will see how to plot this in Maple.
- There are two other plots to consider- x versus t, and y versus t. We will see how to plot these in Maple.

We note that, if $y = 0$, then $\dot{x} = 2x(1 - \frac{1}{2}x)$, which we studied in the last lab, and if $x = 0$, then $\dot{y} = 3y(1 - \frac{1}{3}y)$, which is again what we studied in the last lab.

We can interpret this to mean that, if one species dies out, the other species' population will tend to an equilibrium.

The larger question is: What happens to the populations when they start to compete? Our model assumes that the rate at which the populations change begins to decline when there is an interaction (the xy term measures the effects of species interactions).

2. Analysis in Maple

(1) Draw the phase plot, y versus x with direction arrows. Notice that this is similar to a direction field in that, for each number y and each number x, we can compute $\dot{x}$ (the horizontal rate of change) and $\dot{y}$ (the vertical rate of change). In Maple, first define the differential equations, then use `DEplot`.

```
with(DEtools):
de1:=diff(x(t),t)=2*x(t)*(1-0.5*x(t))-x(t)*y(t);
de2:=diff(y(t),t)=3*y(t)*(1-(1/3)*y(t))-2*x(t)*y(t);

#The following should all be on one line:
DEplot([de1,de2],[x(t),y(t)],t=-1..6,x=0..4,y=0..4,
```

```
[[x(0)=0.3,y(0)=1],[x(0)=2,y(0)=1],[x(0)=2,y(0)=3]],stepsize=0.05,
linecolor=black);
```

This last command plots three sample solutions. How should we understand this picture? (We'll discuss in class)

(2) Draw a plot of x versus t and y versus t using a couple of different initial conditions. Color x blue and y red, and put them on one graph:

```
A:=DEplot([de1,de2],[x(t),y(t)],t=0..15,x=0..4,y=0..4,
[[x(0)=2,y(0)=2.5]],stepsize=0.05,scene=[t,x],linecolor=blue):

B:=DEplot([de1,de2],[x(t),y(t)],t=0..15,x=0..4,y=0..4,
[[x(0)=2,y(0)=2.5]],stepsize=0.05,scene=[t,y], linecolor=red):

with(plots):
display({A,B},title="Competing Species");
```

Now do the same thing (copy and paste) to plot $x(t)$ and $y(t)$ for the initial conditions: $x(0) = 2$, $y(0) = 2.8$. What happens?

(3) Find the equilibrium solutions:

```
eqns:={2*x*(1-(1/2)*x)-x*y=0,3*y*(1-(1/3)*y)-x*y=0};
solve(eqns,{x,y});
```

We can now discuss the predictions of our model: Our model predicts that, given almost any set of initial conditions, one of the populations will die off, and the other will go to an equilibrium. There are a few exceptions: If the populations are $1, 1$ respectively, then they stay in equilibrium. However, these are unstable in the sense that if the population strays from these values, one will die off.

3. Lab 3: A Model of Predator-Prey Interaction

The model we will consider is:

$$\begin{aligned} \dot{x} &= x(1 - \tfrac{K}{10}x) - xy \\ \dot{y} &= -y + xy \end{aligned}$$

We want to explore the solutions to this system of differential equations, and how the solutions depend on the value of K.

3.1. Introductory Questions.

(1) Consider $\dot{x}$ if $y(t) = 0$. What does the population do?
(2) Consider $\dot{y}$ if $x(t) = 0$. What does the population do?
(3) In light of the previous questions, which of $x(t)$, $y(t)$ denotes predator population, and which is prey?

Comment: The xy term denotes how the population changes in the face of predator-prey interactions. Note how one population grows, the other declines.

3.2. The Case Where $K = 0$.

(1) Find the equilibria.
(2) Plot the phase plane, and include two initial conditions.
(3) Plot x versus t and y versus t on the same graph using one of the initial conditions from the phase plane. Are the solutions slightly "out of phase"? Does that make sense?

3.3. **The General Case.**

(1) Find the equilibria (they'll depend on K).
(2) Discuss the bifurcation that occurs when K becomes (slightly) positive. Explain what happens now to the populations- you might want to use graphs to assist the explanation.
(3) As K continues to increase, a second bifurcation occurs. Try to estimate as accurately as you can, the value of K, and describe what happens to the populations before and after this bifurcation. Does disaster befall one of them? (That was a hint!)
(4) The predator-prey model is very similar to the competing species model (in the form of the ODEs). How could you change a "predator-prey" system into a "competing species" system?