UNDERSTANDING $y'' + by' + cy$

1. Introduction

We begin this lab with a theorem:

> Any n^{th} order differential equation
> $$y^{(n)} = f(t, y, y', \ldots, y^{n-1})$$
> can be converted into and analyzed as a system of n first order equations.

We will illustrate this theorem by converting the second order linear equation into a system of first order equations, then we will analyze the behavior of the solutions in these terms.

First, we convert[1] $y'' + by' + cy = 0$ into a system of first order equations. We do this by defining new variables. Let

$$u = y \qquad v = y'$$

Then we write the system of differential equations for u, v:

$$\begin{aligned} \dot{u} = &\quad y' = v \\ \dot{v} = &\quad y'' = -by' - cy = -bv - cu \end{aligned}$$

Therefore, the solutions to $y'' + by' + cy = 0$ are equivalent to the solutions to the system:

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -bv - cu \end{aligned}$$

Some notes:

(1) If we want $y(t)$, all we really need out of the system is $u(t)$.
(2) The phase plane will be a plot of (u, v), or position **and velocity**.
(3) We could get analytic solutions from the linear system, but that would require some linear algebra.
(4) In problems from physics, $b \geq 0$ and $c \geq 0$. However, we can examine the solutions for all b and all c.

2. The Lab

The purpose of this lab is to analyze the behavior of solutions to $y''+by'+cy = 0$ in a different way than you've seen in Differential Equations class- we will study this using a system of equations, and analyze the corresponding phase planes.

We know that to solve $y'' + by' + cy = 0$, we solve the characteristic equation,

$$r^2 + br + c = 0 \Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Therefore, the critical quantity to analyze is $b^2 - 4c$, which we will call the discriminant, Δ. If $\Delta > 0$, there are two real, distinct roots to the characteristic equation.

[1] Your textbook writes this equation as $ay'' + by' + cy = 0$. But, since we assume $a \neq 0$, we divided both sides by a to get the simplified equation, $y'' + By' + Cy = 0$

If $\Delta = 0$ there is a single root, and if $\Delta < 0$, there are complex solutions to the characteristic equation. In ODE class, you learned how to write the solution in each case. In Calc Lab, we will learn how to perform a "qualitative analysis".

To summarize the material we will discuss, you should draw by hand, on one page, the parabola $b^2 - 4c = 0$, where the "$x-$axis" is b, and the "$y-$axis" is c. So, in addition to the four quadrants, we have subdivided the plane into points inside the parabola, on the parabola, and outside the parabola. To keep these regions more concrete, let's focus on some particular values:

- Along the $c-$axis: $b = 0, c = 1$. (In this case, $\Delta < 0$)
- Inside the parabola: $b = 1, c = 1$. (In this case, $\Delta < 0$)
- On the parabola: $b = 2, c = 1$. (In this case, $\Delta = 0$)
- Outside the parabola: $b = 3, c = 1$. (In this case, $\Delta > 0$)
- Along the $b-$axis: $b = 3, c = 0$. (In this case, $\Delta > 0$)

On your piece of paper, plot these points.

2.1. **Lab Questions.**

(1) For each point listed above,
  (a) Find the equilibria
  (b) Plot the corresponding phase plane in Maple, together with some solutions (you may choose the initial conditions).
  (c) Draw a "typical" phase plot on your paper in the appropriate place.

(2) Continue this exploration at some other values of (b, c): What happens if $b = 0$ and $c = 0$?

(3) Why did we say that for "physical" problems, $b > 0$ and $c > 0$? (Consider what the solutions do in the other quadrants).

(4) The form $y'' + by' + cy$ is especially nice if y is an exponential function, a polynomial, sines and cosines, or a product of exponentials with polynomials or exponentials with sines and cosines. Write down the result of using this differential operation on:
  (a) $y(t) = \mathrm{e}^{-2t}\left(\sin(3t) + 2\cos(3t)\right)$
  (b) $y(t) = 3t^2 + 2$
  (c) $y(t) = \cos(2t)$

  Given this experience, what is the form of the particular part of the solution if $y'' + by' + cy = 3t^2 + t$?

(5) In the next question we set $c = 1$, and add a periodic forcing function:
$$y'' + by' + y = \sin(t)$$
  What happens to the solution as $b \to 0$? Use any visualization technique- it might be useful to set $y(0) = 1$ and $y'(0) = 1$. You should consider the homogeneous part of the solution separately from the particular part of the solution (until $b = 0$).