LAB 5: Power Series Solutions to ODEs

1. Introduction

A great technique in solving differential equations is to change the differential equation into one (or many) *algebraic* equation(s), so that the solution to the algebraic equation will give the solution to the differential equation.

Our ansatz is that y is an analytic function (representable as a power series). The expansion of y about an initial time, $t = a$ can be written as:

$$y(t) = \sum_{n=0}^{\infty} c_n (t-a)^n = c_0 + c_1(t-a) + c_2(t-a)^2 + \dots$$

(Note that $y(a) = c_0$) and

$$y'(t) = \sum_{n=1}^{\infty} n c_n (t-a)^{n-1} = c_1 + 2c_2(t-a) + 3c_3(t-a)^2 + \dots$$

(Note that $y'(a) = c_1$) and

$$y''(t) = \sum_{n=2}^{\infty} n(n-1) c_n (t-a)^{n-2} = 2c_2 + 3 \cdot 2 \cdot c_3(t-a) + \dots$$

Note that in general, we can define all the coefficients in terms of the initial conditions $c_0 = y(a)$ and $c_1 = y'(a)$- these will be the two unknowns (until we insert a set of initial conditions).

Now, if we have some differential equation like:

$$y'' = y$$

then

$$\sum_{n=2}^{\infty} n(n-1)c_{n-2}(t-a)^{n-2} = \sum_{n=0}^{\infty} c_n(t-a)^n \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_{n-2}(t-a)^{n-2} - \sum_{n=0}^{\infty} c_n(t-a)^n = 0$$

We'd like to write this as a single sum. Let's write out the first few terms of the resulting polynomial to see if we can find the pattern:

$$(2c_2 - c_0) + (3 \cdot 2 \cdot c_3 - c_1)(t-a) + (4 \cdot 3 \cdot c_4 - c_2)(t-a)^2 + \dots$$

From this, we see that the general term (with n starting at 0) would be:

$$\left((n+2)(n+1)c_{n+2} - c_n\right)(t-a)^n$$

so overall, we get:

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)c_{n+2} - c_n\right)(t-a)^n = 0$$

This means that:

$$(n+2)(n+1)c_{n+2} = c_n$$

Writing out the first few terms:

$$\begin{aligned} c_2 &= \tfrac{1}{2} c_0 \\ c_3 &= \tfrac{1}{3 \cdot 2} c_1 \\ c_4 &= \tfrac{1}{4 \cdot 3} c_2 = \tfrac{1}{4 \cdot 3 \cdot 2} c_0 \\ c_5 &= \tfrac{1}{5 \cdot 4} c_3 = \tfrac{1}{5 \cdot 5 \cdot 4 \cdot 3 \cdot 2} c_1 \end{aligned}$$

Thus we see that the power series is generating the exponential function, which is what we expected. In general, it is not possible to always get a "closed form" of the solution.

2. The Lab

We will read over the Maple commands and examples from the website to see how to get and analyze power series solutions to ODEs. Once you've done that, write up the solutions to the following problems, including any relevant graphs. You'll also turn in a printout of the Maple worksheet(s) (I don't need to see the Maple output, just your commands).

(1) Let
$$\frac{dy}{dt} - 2y = \sin(y), \quad y(0) = 1$$
  (a) Notice that this is an autonomous first order equation. Give the phase plot- if there are equilibria, find them.
  (b) Try to get an exact solution- Can Maple find one?
  (c) Have Maple give a series solution of order 12 (the polynomial part will have degree 11). Plot the series solution and the direction field. Comment on the accuracy of the solution.

(2) Let
$$(2 + x^2)y'' - xy' + 4y = 0, \quad y(0) = -1, y'(0) = 3$$
  (a) Have Maple give a series solution of order 12. Convert this into a polynomial, and try inserting it into the differential equation. What is the result? Explain.
  (b) Determine the recurrence relation for the coefficients.
  (c) Plot several partial sums of the series solution together on the same graph. Estimate the interval on which your best solution is reasonably accurate.

(3) Consider the differential equation in the previous problem. Maple's `powseries` commands don't allow us to use any other initial time but $x_{\text{init}} = 0$. How would I convert the following IVP so that I can use the `powseries` commands?
$$(2 + x^2)y'' - xy' + 4y = 0, \quad y(2) = -1, y'(2) = 1$$
Do it, and verify your solution by also using `dsolve` and compare the series.

(4) Let
$$\cos(x)y'' + xy' - 2y = 0$$
  (a) Can Maple determine an exact solution?
  (b) If $y(0) = 0$ and $y'(0) = 1$, find a series solution of order 12 and plot it in an interval about the origin.
  (c) In the general case (no initial conditions), find the first four nonzero terms in each of two linearly independent power series solutions about the origin.