LAB 6 GOALS: LAPLACE TRANSFORMS AND ODES

1. Pre Lab 6 Goals:

(1) Learn the definitions of three new functions:
- The Laplace Transform:
$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt$$
- The Heaviside Function:
$$H(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{otherwise} \end{cases}$$
- The Dirac $\delta-$function. Not a regular function- called a "generalized function". We can define it using a sequence of functions. If
$$F_n(t) = \begin{cases} \frac{n}{2} & -\frac{1}{n} \leq t \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$
then $\delta(t) = \lim_{n\to\infty} F_n(t)$. Then $\delta(t)$ has the following properties:
$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$
And.
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad \int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a)$$
- Intuitively, we think of the "derivative" of $H(t)$ to be $\delta(t)$ (so, the "antiderivative" of $\delta(t)$ is $H(t)$).

(2) Learn the new Maple commands:
- `with(inttrans):` and `readlib(laplace):`
- `laplace()` and `invlaplace()`
- `Heaviside(t), Dirac(t)`
- `dsolve(IVP, y(t), method=laplace);`

2. Lab 6 Goals:

(1) Use the Laplace Transform (and Maple!) to solve second order linear differential equations with forcing functions (that may not be continuous in the usual sense).
(2) Understand where Laplace Transforms fit into the larger picture of differential equations.
(3) Be able to state alternatives to using the Laplace transform, where appropriate.
(4) Understand the relationships between the forcing functions and the response functions when working with the Dirac function.
(5) Use and manipulate damped oscillations.

3. Pre Lab 6 (In Class)

- Definition: The Laplace Transform of $y(t)$ is:

$$Y(s) = \int_0^\infty \mathrm{e}^{-st} y(t)\, dt = \mathcal{L}(y)$$

  Notes:
  - The integral is a limit. For the limit to exist, $y(t)$ must be piecewise continuous and be "of exponential order" (bounded by a function of the form $M\mathrm{e}^{kt}$).
  - The Laplace Transform of $y(t)$ is a function of s.
  - Normally, we would use "integration by parts" to actually compute the Laplace Transform.
  - The Laplace Transform is invertible "most of the time".
- The Laplace Transform in Maple, by hand:
  Example: Compute the Laplace Transform of t^3, e^{-2t}, $\sin(at)$.

```
assume(s>0);
F:=int(t^3*exp(-st),t=0..infinity);
G:=int(exp(-2*t)*exp(-s*t),t=0..infinity);
H:=int(sin(a*t)*exp(-s*t),t=0..infinity);
```

- The Laplace Transform in Maple, using built-in commands:

```
s:='s';
with(inttrans):
readlib(laplace):
F:=laplace(t^3,t,s);
G:=laplace(exp(-2*t),t,s);
H:=laplace(sin(a*t),t,s);
```

- Check your previous answers by inverting the transforms:

```
f:=invlaplace(F,s,t);
g:=invlaplace(G,s,t);
h:=invlaplace(H,s,t);
```

- Let $Y(s)$ be the Laplace Transform of $y(t)$. That is, $\mathcal{L}(y) = Y(s)$. In Differential Equations class, it is shown that,

$$\mathcal{L}(y'(t)) = Y(s) - y(0), \text{ and } \mathcal{L}(y''(t)) = Y(s) - sy'(0) - y(0)$$

  In particular, we can apply the Laplace Transform to a differential equation.
  It would be meaningless to do this unless the transform is invertible. In fact, if the domain of $\mathcal{L}$ is the set of continuous functions, then $\mathcal{L}$ is invertible.
- Practice Problems: Have Maple take the Laplace transform of the following differential equations, solve for $Y(s)$, then invert the transform- we're solving the differential equations algebraically!
  (1) $y'' - 2y' + 2y = \cos(t)$
  (2) $y'' + 2y' + y = 4\mathrm{e}^{-t}$

```
deqn1:=diff(y(t),t$2)-2*diff(y(t),t)+2*y(t)=cos(t);
eqn1:=laplace(deqn,t,s);
eqn2:=solve(eqn1,laplace(y(t),t,s));
invlaplace(eqn2,s,t);
```

```
deqn2:=diff(y(t),t$2)+2*diff(y(t),t)+y(t)=4*exp(-t);
eqn1:=laplace(deqn,t,s);
eqn2:=solve(eqn1,laplace(y(t),t,s));
invlaplace(eqn2,s,t);
```

- Of course, Maple can do this automatically:

```
dsolve(deqn1,y(t),method=laplace);
dsolve(deqn2,y(t),method=laplace);
```

- Our two new functions: Heaviside and Dirac.
  Maple has these built-in as: `Heaviside(t)` and `Dirac(t)`
  (1) The Heaviside function is used like an "ON-OFF" switch (and that's literal when we do modeling!). In Maple, plot the following functions:
  $$H(t-2) - H(t-4) \qquad H(t-2) * H(4-t)$$
  where H is the Heaviside function. What do you notice? If you want to model a switch that goes ON at time A and OFF at time B, you can write either $H(t-a) - H(t-b)$ or $H(t-a) * H(b-t)$
  (2) In Maple, take the derivative of the Heaviside function- what do you get?
  (3) Have Maple compute the Laplace transform of the Heaviside function, $H(t-2) - H(t-4)$.
  (4) Have Maple compute the Laplace transform of the Dirac $\delta-$function.
  (5) We use the Dirac $\delta-$function to model extremely swift events, like hitting a pendulum with a hammer. For example, suppose we have a pendulum with some resistance and it is currently at rest. At $t = 4$, bang it with a hammer.
  $$y'' + \frac{1}{2}y' + y = \delta(t-4), \quad y(0) = 0, y'(0) = 0$$
  Solve the ODE in Maple and plot the result. Once you've done that, bang it again in the opposing direction at $t = 10$. Re-model, re-solve, and re-plot. Do another hit at $t = 15$, re-model, re-solve, and re-plot. Do you see the pattern in the algebraic solution?
  Note: In Maple, the ODE is:

```
deqn:=diff(y(t),t$2)+(1/2)*diff(y(t),t)+y(t)=Dirac(t-4);
```

- Damped oscillations. It is common, with the second order linear ODE, to deal with damped oscillations- that is, functions of the form:
  $$\mathrm{e}^{-kt}(A\sin(mt) + B\cos(mt))$$
  For example, plot the three functions on the same graph:
  $$\mathrm{e}^{-2t}, \quad \sin(3t), \quad \mathrm{e}^{-2t}\sin(3t)$$
  Explain what you see.