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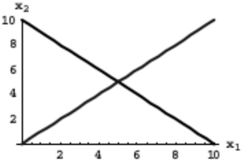
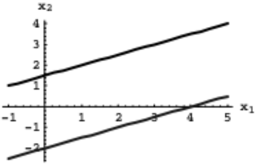
Consider Systems of Two Variables

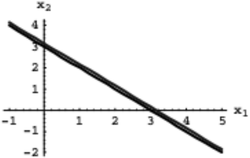
$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$

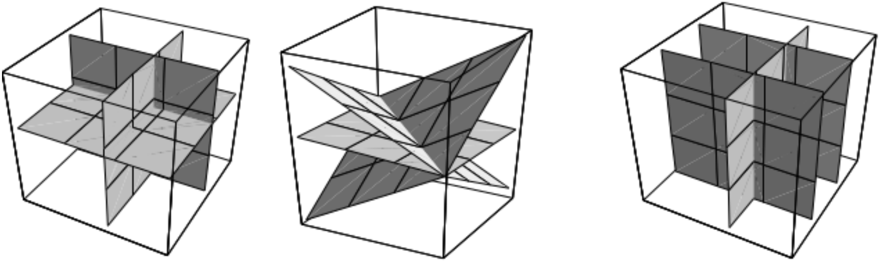
$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$







Consider Systems of Three Variables



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For the rest of today, we discuss an algorithm for solving a linear system.

Strategy for Solving a System

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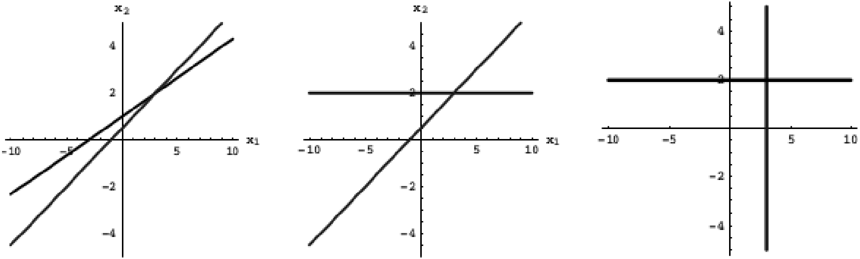
Def: Two systems are **equivalent** if they have the same solution set.

Example

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \quad \longrightarrow \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \quad \longrightarrow \quad \begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array}$$



Matrix Notation

Definition: The system to the left is equivalent to the "augmented" matrix to the right:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \Leftrightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

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Example: Convert the three systems from the previous example into the equivalent augmented matrices:

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- (Replacement) Add one row to a multiple of another row. $r_i + kr_j \to r_i$
- (Interchange) Interchange two rows $r_i \leftrightarrow r_j$
- (Scaling) Multiply all entries in row by a non-zero scalar. $kr_i \to r_i$.

Example

Given the matrices below, what operation was performed?

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

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$$R_3 + 4R_1 \to R_3$$

Continuing:

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

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$(1/2)R_2 \rightarrow R_2$. And the next one:

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$R_3 + 3R_2 \rightarrow R_3$.

And the two row operations to get:

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

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$R_2 + 4R_3 \rightarrow R_2$ and $R_1 + (-1)R_3 \rightarrow R_1$. And finally:

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

In this case, we read off the solution:

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$$x_1 = 29, x_2 = 16, x_3 = 3$$

Two Fundamental Questions: Existence and Uniqueness

- Is a given system **consistent** (does a solution exist)? (This is "existence")
- If a solution exists, is it unique? (Infinite number, or only one?)

In the last example, it would suffice to have the system in "trangular form":

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Why?

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Why?

In the third equation, we set $x_3 = 3$, and substitute that into equations 1 and 2:

$$\begin{aligned} x_1 - 2x_2 + (3) &= 0 \\ x_2 - 4(3) &= 4 \end{aligned}$$

From which $x_2 = 16$, so that $x_1 = 29$ (this is backsubstitution).

Consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

To assist you, consider the matrices produced by row ops:

$$\left[\begin{array}{rrr|r} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

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The last equation is never true: $0 = -3$, so INCONSISTENT.

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For what value(s) of h will the following be consistent?

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If $h + 8/3 \neq 0$, the system is INCONSISTENT.