6.7: Inner Product Spaces

All of the geo. props we discussed for $\mathbb{R}^n$ (length, distance, orthogonality), came to us from the dot product.

Is there something like the dot product, but for other types of vector spaces? Yes- It's called an **inner product**.

Definition: Let V be a vector space, and let u, v, w be vectors in V and c, d be any scalars. Then an **inner product** is a function from $V \times V$ to $\mathbb{R}$ and denoted by:

$$\langle u, v \rangle$$

with the following properties:

- $\langle u, v \rangle = \langle v, u \rangle$
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle cu, v \rangle = c\langle u, v \rangle$
- $\langle u, u \rangle \geq 0$, and $\langle u, u \rangle = 0$ iff $u = 0$.

Example 1

Let u, v be vectors in $\mathbb{R}^2$, and define

$$\langle u, v \rangle = 4u_1v_1 + 5u_2v_2$$

(We might think of this as a "weighted dot product"). Show that this is an inner product.

Example 2

Let $t_0, t_1, \ldots, t_n$ be $n+1$ distinct numbers. For vectors $p, q \in P_n$, define the inner product by first evaluating p, q at the $n + 1$ points (so that we have two vectors in $\mathbb{R}^{n+1}$. Then define:

$$\langle p, q \rangle = \vec{p} \cdot \vec{q}$$

Then items 1-3 are straightforward. What about (4)? Well, first note that

$$\langle p, p \rangle = (p(t_0))^2 + (p(t_1))^2 + \ldots + (p(t_n))^2 \geq 0$$

Note that if $\langle p, p \rangle = 0$, then $p(t_i) = 0$ for $i = 0, 1, \ldots, n+1$. A polynomial of degree $\leq n$ that is zero at $n+1$ points must be the zero function.

Example 3: $C[a, b]$

Let $f, g \in C[a, b]$. Show that the following function defines an inner product:

$$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$$

Items 1-3 again are easy to show. What about (4)? In particular, think about this- If the integral of $f^2(t)$ is zero, does that mean that $f(t) = 0$? (Yes, as long as f is continuous)

Geometry of an Inner Product Space

Once a vector space is given an inner product, then we can define length, distance and angle just as before:

$$\|f\| = \sqrt{\langle f, f\rangle}$$

$$\text{dist}(f, g) = \|f - g\|$$

And finally, we define θ, the angle between f and g as the angle satisfying:

$$\cos(\theta) = \frac{\langle f, g\rangle}{\|f\|\|g\|}$$

Therefore, we also say that f, g are orthogonal (with respect to the given inner product) if $\langle f, g\rangle = 0$.

Two Inequalities

There are also two important inequalities that inner products must satisfy:

- The Cauchy-Schwartz inequality:
$$|\langle u, v\rangle| \leq \|u\|\,\|v\|$$
(You can see how this might stem directly from our definition of θ)
- The triangle inequality:
$$\|f + g\|^2 \leq \|f\|^2 + \|g\|^2$$
For a proof, recall that
$$\langle f + g, f + g\rangle = \langle f, f\rangle + 2\langle f, g\rangle + \langle g, g\rangle$$
Then use the Cauchy-Schwartz inequality on the middle term.

Applications

Now we can perform projections. For example, given that the inner product on $C[0,1] = \int_0^1 f(t)g(t)\,dt$, how would we project $f(t) = t$ onto $g(t) = 1 + t^2$?

$$\text{Proj}_{g(t)}(f(t)) = \left(\frac{\langle f, g\rangle}{\langle g, g\rangle}\right) g(t)$$

so we would need to compute:

$$\langle f, g\rangle = \int_0^1 t(1 + t^2)\,dt = \frac{3}{4} \qquad \langle g, g\rangle = \int_0^1 (1 + t^2)^2\,dt = \frac{28}{15}$$

so that simplifying we get

$$\frac{3/4}{28/15}(1 + t^2) = \frac{45}{112}(1 + t^2)$$

A Basis for Functions

We have defined that an analytic function is any function that is equal to its Taylor series (based at x_0). This means that the set of functions

$$\left\{1, (x - x_0), (x - x_0)^2, (x - x_0)^3, \cdots\right\}$$

will form a **basis** for the space of such functions. Similarly, the set of monomials forms a basis for $C[-1, 1]$, and if we further set the inner product as:

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt$$

then we can construct a set of **orthogonal polynomials**. So let's do that using Gram-Schmidt (without normalization):

$$P_0(x) = 1$$

Then

$$P_1(x) = x - \mathrm{Proj}_1(x) = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 = x - 0 = x$$

Now,

$$P_2(x) = x^2 - \mathrm{Proj}_x(x^2) - \mathrm{Proj}_1(x^2) = x^2 - \frac{\langle x^2, x\rangle}{\langle x, x\rangle} x - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1$$

where

$$\langle 1, 1\rangle = 2 \qquad \langle x, x^2\rangle = 0 \qquad \langle x^2, 1\rangle = \frac{2}{3}$$

so that

$$P_2(x) = x^2 - \frac{1}{3}$$

and so on...

The set of these polynomials is called the **Legendre polynomials**.

As a side note, the Laguerre polynomials are a set of polynomials that are orthogonal on $[0, \infty)$ with the inner product

$$\langle f, g\rangle = \int_{0}^{\infty} f(t)g(t)\mathrm{e}^{-t}\, dt$$