Math 240, Second Exam REVIEW QUESTIONS

1. Short Answer:

   (a) Finish the definition: The set of vectors $\{v_1, \ldots, v_k\}$ **spans** set V if:

   (b) Finish the definition: The set of vectors $\{v_1, \ldots, v_k\}$ for a **basis** for vector space V if:

   (c) Finish the definition: The **rank** of a matrix is:

   (d) How was the matrix-matrix product AB defined?

   (e) Finish the definition:
   The $n \times n$ matrix A is **invertible** if:
   (Note that this is the definition, not something equivalent to the definition).

   (f) If A is an $m \times n$ matrix, the column space of A is a subspace of $\mathbb{R}^?$, and it is defined as:

   (g) If A is an $m \times n$ matrix, the null space of A is a subspace of $\mathbb{R}^?$ and it is defined as:

   (h) Finish the definition:
   Subset H in vector space V is a **subspace** if:

   (i) Find the inverse of $\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$

2. Find the inverse of the matrix A below:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

3. Suppose A, B and X are $n \times n$ matrices, with A, X, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B$$

   First, explain why B is invertible, then solve the equation for X. If you need to invert a matrix, explain why it is invertible.

4. Show that, if AB is invertible, then so is A (assume A, B are $n \times n$).

5. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.
   If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

6. Assume that A and B are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) State which vector space contains each of the four subspaces, and state the dimension of each of the four subspaces:

   (b) Find a basis for Col(A):

   (c) Find a basis for Row(A):

   (d) Find a basis for Null(A):

7. Determine if the following sets are subspaces of V. Justify your answers.

- $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a \geq 0, \quad b \geq 0, \quad c \geq 0 \right\}, \qquad V = \mathbb{R}^3$
- $H = \left\{ \begin{bmatrix} a+3b \\ a-b \\ 2a+b \\ 4a \end{bmatrix}, \quad a, b \text{ in } \mathbb{R} \right\}, \quad V = \mathbb{R}^4$
- $H = \{f : f'(x) = f(x)\}, V = C^1(-\infty, \infty)$
  (C^1 is the space of differentiable functions where the derivative is continuous).
- H is the set of vectors in $\mathbb{R}^3$ whose first entry is the sum of the second and third entries, $V = \mathbb{R}^3$.

8. Prove that, if $T : V \mapsto W$ is a linear transformation between vector spaces V and W, then the range of T, which we denote as $T(V)$, is a subspace of W.

9. Let H, K be subspaces of vector space V. Define $H+K$ as the set below, and see if $H+K$ is a subspace (check all parts of the definition).

$$H + K = \{\mathbf{w} \mid \mathbf{w} = \mathbf{u} + \mathbf{v}, \text{ for some } \mathbf{u} \in H, \mathbf{v} \in K\}$$

10. Let A be an $n \times n$ matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement "A is invertible". Use the following concepts, one in each statement: (a) Null(A) (b) Basis (c) Rank (d) det(A)

11. Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

12. Show that $\{1, 2t, -2 + 4t^2\}$ is a basis for P_2.

13. Let $T : V \to W$ be a $1-1$ and linear transformation on vector space V to vector space W. Show that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are linearly dependent vectors in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent vectors in V.

14. Use Cramer's Rule to solve the system:

$$\begin{array}{rrrl} 2x_1 & +x_2 & & = 7 \\ -3x_1 & & +x_3 & = -8 \\ & x_2 & +2x_3 & = -3 \end{array}$$

15. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$, and $\mathbf{w} = [2, 1]^T$. Is $\mathbf{w}$ in the column space of A? Is it in the null space of A?

16. Prove that the column space is a vector space using a very short proof, then prove it directly by showing the three conditions hold.

17. If A, B are 4×4 matrices with $\det(A) = 2$ and $\det(B) = -3$, what is the determinant of the following (if you can compute it): (a) $\det(AB)$, (b) $\det(A^{-1})$, (c) $\det(5B)$
(d) $\det(3A - 2B)$, (e) $\det(B^T)$

18. True or False, and give a short reason:

    (a) If $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.

    (b) Let A be $n \times n$. Then $\det(A^T A) \geq 0$.

(c) If A^3 is the zero matrix, then $\det(A) = 0$.

(d) $\mathbb{R}^2$ is a two dimensional subspace of $\mathbb{R}^3$.

(e) Row operations preserve the linear dependence relations among the rows of A.

(f) The sum of the dimensions of the row space and the null space of A equals the number of rows of A.

19. Let the matrix A and its RREF, R_A, be given as below:

$$A = \begin{bmatrix} 1 & 1 & 7 & 2 & 2 \\ 3 & 0 & 9 & 3 & 4 \\ -3 & 1 & -5 & -2 & 3 \\ 2 & 2 & 14 & 4 & 2 \end{bmatrix} \qquad R_A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the columns of A are $\mathbf{a}_1, \cdots, \mathbf{a}_5$.

Similarly, define Z and its RREF, R_Z, as:

$$Z = \begin{bmatrix} 4 & 5 & 3 & 4 \\ 5 & 6 & 5 & -3 \\ 10 & -3 & 9 & -106 \\ 4 & 10 & 2 & 44 \end{bmatrix} \qquad R_z = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Label the columns of Z as $\mathbf{z}_1, \cdots, \mathbf{z}_4$.

(a) Find the rank of A and a basis for the column space of A (use the notation $\mathbf{a}_1$, etc.). Similarly, do the same for Z:

(b) You'll notice that the rank of A is the rank of Z. Here is a row reduction using some columns of A and Z:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 5 & 3 \\ 3 & 0 & 4 & 5 & 6 & 5 \\ -3 & 1 & 3 & 10 & -3 & 9 \\ 2 & 2 & 2 & 4 & 10 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Are the subspaces spanned by the columns of A and Z equal?

(c) Let $\mathcal{B}$ and be the set of basis vectors used for the column spaces of A found in (a). Find the change of coordinates matrix $P_{\mathcal{B}}$ that changes the coordinates from $\mathcal{B}$ to the standard basis, then find the coordinates of $\mathbf{z}_1$ with respect to $\mathcal{B}$ (Hint: The second part does not rely on the first).

(d) Find the coordinates of $\mathbf{z}_4$ using the basis vectors in $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$.

20. Short Answer:

(a) Define the *kernel* of a transformation T:

(b) Define the *dimension* of a vector space:

(c) We said that $\mathbb{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$. What is the isomorphism?

(d) If C is 4×5, what is the largest possible rank of C?
What is the smallest possible dimension of the null space of C?

(e) If A is a 4×7 matrix with rank 3, find the dimensions of the four fundamental subspaces of A.

(f) Show that the coordinate mapping (from $n-$dimensional vector space V to $\mathbb{R}^n$) is onto.

21. Let A be $m \times n$ and let B be $n \times p$. Show that the $\text{rank}(AB) \leq \text{rank}(A)$. (Hint: Explain why every vector in the column space of AB is in the column space of A).

22. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) If T is one-to-one, what is the dimension of the range of T?

(b) What is the dimension of the kernel of T if T maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$? Explain.

23. Find the determinant of the matrix A below:

$$A = \begin{bmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{bmatrix}$$

24. Let A, B be given below. Form the matrix product AB, if defined:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$$

25. Given the matrix A, B below:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) Compute only the $(2, 3)$ entry of AB:

(b) Compute only the $(3, 2)$ entry of AB^T:

(c) Compute $B - 3I_3$:

(d) Compute C_{23} for matrix A (that's the (2,3) cofactor).

26. If A is the 2×3 matrix below, find a matrix C so that $AC = I$, but note that C is not the inverse of A. To simplify your computations, I've given you one form for C that you might use.

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ 0 & 0 \end{bmatrix}$$

27. Suppose A is $n \times n$ with the property that $A\mathbf{x} = \vec{0}$ has only the trivial solution. Without using the invertible matrix theorem, explain directly why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for every $\mathbf{b}$.

28. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent. (Hint: You might think about whether or not A^2 must be invertible).

29. Suppose subspace H is the span of the two vectors below in set $\mathcal{B}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2\}$$

(a) Does $\mathcal{B}$ span $\mathbb{R}^3$? Why or why not?

(b) Find $[\mathbf{v}_1]_{\mathcal{B}}$

(c) If $\mathbf{c} = (3, 3, 0)$, find $[\mathbf{c}]_{\mathcal{B}}$