

on vector spaces of functions, and Chapter 4 extends the theory of vectors in \mathbb{R}^n to include such functions. Later on,

you will see how other vector spaces arise in engineering, physics, and statistics.

WEB

The mathematical seeds planted in Chapters 1 and 2 germinate and begin to blossom in this chapter. The beauty and power of linear algebra will be seen more clearly when you view \mathbb{R}^n as only one of a variety of vector spaces that arise naturally in applied problems. Actually, a study of vector spaces is not much different from a study of \mathbb{R}^n itself, because you can use your geometric experience with \mathbb{R}^2 and \mathbb{R}^3 to visualize many general concepts.

Beginning with basic definitions in Section 4.1, the general vector space framework develops gradually throughout the chapter. A goal of Sections 4.3–4.5 is to demonstrate how closely other vector spaces resemble \mathbb{R}^n . Section 4.6 on rank is one of the high points of the chapter, using vector space terminology to tie together important facts about rectangular matrices. Section 4.8 applies the theory of the chapter to discrete signals and difference equations used in digital control systems such as in the space shuttle. Markov chains, in Section 4.9, provide a change of pace from the more theoretical sections of the chapter and make good examples for concepts to be introduced in Chapter 5.

4.1 VECTOR SPACES AND SUBSPACES

Much of the theory in Chapters 1 and 2 rested on certain simple and obvious algebraic properties of \mathbb{R}^n , listed in Section 1.3. In fact, many other mathematical systems have the same properties. The specific properties of interest are listed in the following definition.

DEFINITION

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.¹ The axioms must hold for all vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} in V and for all scalars c and d .

1. The sum of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} + \mathbf{v}$, is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each \mathbf{u} in V , there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of \mathbf{u} by c , denoted by $c\mathbf{u}$, is in V .
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

¹Technically, V is a *real vector space*. All of the theory in this chapter also holds for a *complex vector space* in which the scalars are complex numbers. We will look at this briefly in Chapter 5. Until then, all scalars are assumed to be real.