Linear Algebra- Final Exam Review Questions

These are meant to give you a sample of questions cutting across topics. Be sure you've looked over your old exams as well!

1. Let A be invertible. Show that, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, so are $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$. NOTE: It should be clear from your answer that you know the definition of linear independence.

2. Find the line of best first for the data:

x	0	1	2	3
y	1	1	2	2

3. Let $A = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$. Diagonalize A, if possible.

4. Let V be the vector space spanned by the vectors (which are functions in this case):

$$f_1(x) = x\sin(x) \qquad f_2(x) = x\cos(x) \qquad f_3(x) = \sin(x) \qquad f_4(x) = \cos(x)$$

   Define the operator $D : V \to V$ as the derivative, so that $Df_1 = f_1'(x)$, for example.

   (a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4

   (b) Find the eigenvalues of A.

   (c) Is the matrix A diagonalizable?

5. Short answer:

   (a) Let H be the subset of vectors in $\mathbb{R}^3$ consisting of those vectors whose first element is the sum of the second and third elements. Is H a subspace?

   (b) Explain why the image of a linear transformation $T : V \to W$ is a subspace of W

   (c) Is the following matrix diagonalizable? Explain. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$

   (d) If the column space of an 8×4 matrix A is 3 dimensional, give the dimensions of the other three fundamental subspaces. Given these numbers, is it possible that the mapping $\mathbf{x} \to A\mathbf{x}$ is one to one? onto?

6. True or False, and give a short reason:

   (a) If A is 3×3, then $\det(5A) = 5\det(A)$.

   (b) If A, B are $n \times n$ with $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.

   (c) If A is $n \times n$ and $\det(A) = 2$, then $\det(A^3) = 6$.

   (d) If B is produced by taking row 1 and A and adding 3 times row 3, then putting the result back in row 1, then $\det(B) = 3\det(A)$.

7. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, then

   (a) Find the solution to $A\mathbf{x} = \mathbf{b}$ using Cramer's Rule.

   (b) Find **only** the $(1, 2)$ entry of A^{-1} by using the formula for the adjoint.

8. Find a basis for the null space, row space and column space of A, if $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

9. Find an orthogonal basis using the Gram-Schmidt process if $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

10. Using your answer to 9, if $\mathbf{y} = (1, 0, 0, 0)$, then find a vector in $\hat{\mathbf{y}} \in W$ and a vector in $\mathbf{z} \in W^{\perp}$ so that $\mathbf{y}$ is the sum of the vector in W and $W^{\perp}$.

11. (Referring to the previous problem) What is the distance between $\mathbf{y}$ and W?

12. If $\mathbf{x}_1, \mathbf{x}_2$ are the two vectors in problem 9, find a numerical expression for the angle between them.

13. Let $\mathbb{P}_n$ be the vector space of polynomials of degree n or less. Let W_1 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(0)\mathbf{p}(1) = 0$. Let W_2 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(2) = 0$. Which of the two is a subspace of $\mathbb{P}_n$?

14. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

15. Define $T : P_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

   (a) Find the image under T of $p(t) = 5 + 3t$.

   (b) Show that T is a linear transformation.

   (c) Find the kernel of T. Does your answer imply that T is $1-1$? Onto? (Review the meaning of these words: kernel, one-to-one, onto)

   (d) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2. (This means that the matrix will act on the *coordinates* of p).

16. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ so that $\|\mathbf{v}\| = 1$, and let $Q = I - 2\mathbf{v}\mathbf{v}^T$. Show (by direct computation) that $Q^2 = I$.

17. Let A be $m \times n$ and suppose there is a matrix C so that $AC = I_m$. Show that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$. Hint: Consider $AC\mathbf{b}$.

18. If B has linearly dependent columns, show that AB has linearly dependent columns. Hint: Consider the null space.

19. If λ is an eigenvalue of A, then show that it is an eigenvalue of A^T.

20. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelogram with vertices at $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{u} + v$. Compute the area of S.

21. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.
If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

22. Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix} \right\}$, and $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$. Write down the matrices that take $[x]_C$ to $[x]_B$ and from $[x]_B$ to $[x]_C$.

23. Define an *isomorphism:*

24. Let
$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$
Find at least two $\mathcal{B}-$coordinate vectors for $\mathbf{x} = [1, 1]^T$.

25. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for vector space V. Explain why the $\mathcal{B}-$coordinate vectors of $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are the columns of the $n \times n$ identity matrix:

26. Find the volume of the parallelepiped formed by $\mathbf{0}$, $\mathbf{a}$,$\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}+\mathbf{b}$, $\mathbf{c}+\mathbf{b}$, $\mathbf{c}+\mathbf{a}$, and the sum of all three.
$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

27. Let $\mathbf{u} = (5, -6, 7)$. Let W be the set of all vectors orthogonal to $\mathbf{u}$. (i) Geometrically, what is W? (ii) Compute the projection of $\mathbf{x} = (1, 2, 3)$ onto W. (iii) Write W as the span of some set (that is, find a basis for W).

28. Suppose A is a 3×4 matrix, and any solution to $A\mathbf{x} = \mathbf{0}$ can be written as a linear combination:
$$\mathbf{x} = s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Remembering that A is 3×4, find the row reduced echelon form of A:

(b) Find the dimensions of all four fundamental subspaces: Col(A), Row(A), Null(A), and Null(A^T).

(c) You have enough information to find bases for one or more of these subspaces- Find those bases.

29. Suppose A is a 6×3 matrix and $A\mathbf{x} \neq \mathbf{0}$ if $\mathbf{x} \neq \mathbf{0}$.

(a) What can be said about the columns of A?

(b) Show that $A^TA\mathbf{x} \neq \mathbf{0}$ (for $\mathbf{x} \neq \mathbf{0}$) by explaining this key step:

$$\text{If } A^TA\mathbf{x} = \mathbf{0}, \text{ then clearly } \mathbf{x}^TA^TA\mathbf{x} = 0, \text{ and then (Why?) } A\mathbf{x} = \mathbf{0}.$$

(c) By the previous step, we know that A^TA is invertible (Why?).

30. Consider the system:

$$\begin{aligned} x + 2y - z &= 3 \\ x + 2y - z &= 2 \\ x + 2y - z &= -2 \end{aligned}$$

Clearly, the system is inconsistent. Find the least squares solution, and write the solution in (parametric) vector form.