Linear Algebra- Final Exam Review

1. Let A be invertible. Show that, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, so are $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$. NOTE: It should be clear from your answer that you know the definition.

   SOLUTION: We need to show that the only solution to:

   $$c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + c_3 A\mathbf{v}_3 = 0$$

   is the trivial solution. Factoring out the matrix A,

   $$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = 0$$

   Think of the form $A\hat{\mathbf{x}} = \mathbf{0}$. Since A is invertible, the only solution to this is $\hat{\mathbf{x}} = 0$, which implies that the only solution to the equation above is the solution to

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

   Which is (only) the trivial solution, since the vectors are linearly independent. (NOTE: Notice that if the original vectors had been linearly dependent, this last equation would have non-trivial solutions).

2. Find the line of best fit (sorry for the typo!) for the data:

x	0	1	2	3
y	1	1	2	2

   Let A be the matrix formed by a column of ones and a column containing the data in x. If we write the original system as

   We form the normal equations $A^TA\boldsymbol{\beta} = A^T\mathbf{y}$-

   $$A^TA = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \quad \Rightarrow \quad (A^TA)^{-1} = \frac{1}{20}\begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}$$

   The solution is $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{y} = \frac{1}{10}[4, 9]^T$

3. Let $A = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$. We'll try to diagonlize it by getting the eigenvalues and eigenvectors. The characteristic equation is

   $$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda = -1, 2$$

   We see that we have distinct eigenvalues, so our eigenvectors will be linearly independent (and the matrix will be diagonalizable). For each eigenvalue:

   - $\lambda = -1$, so $(0 - -1)v_1 - v_2 = 0$, and choose $\mathbf{v}_1 = (1, 1)$
   - $\lambda = 2$, so $(0 - 2)v_1 - v_2 = 0$, and choose $\mathbf{v}_2 = (1, -2)$

Now we see that $A = PDP^{-1}$, if

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

4. Let V be the vector space spanned by the functions:

$$f_1(x) = x\sin(x) \qquad f_2(x) = x\cos(x) \qquad f_3(x) = \sin(x) \qquad f_4(x) = \cos(x)$$

Define the operator $D : V \to V$ as the derivative.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4
SOLUTION: Remember that we first look at where the operator sends the basis vectors. In this case, differentiate each of them:

$$Df_1 = \sin(x) + x\cos(x) = f_3 + f_2 \qquad Df_2 = \cos(x) - x\sin(x) = f_4 - f_1$$

$$Df_3 = \cos(x) = f_4 \qquad Df_4 = -\sin(x) = -f_3$$

In terms of the coordinates, then matrix is then:

$$A = [[Df_1]\ [Df_2]\ [Df_3]\ [Df_4]] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(b) Find the eigenvalues of A.
SOLUTION: Take the determinant of $A - \lambda I$. I'll expand it using the first row:

$$\begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} + (1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$(-\lambda)(-\lambda)(\lambda^2+1) + (1)(1)(\lambda^2+1) = (\lambda^2+1)(\lambda^2+1)$$

so that $\lambda = \pm i, \pm i$.

(c) Is the matrix A diagonalizable?
SOLUTION: No, not using real numbers.

5. Short answer:

(a) Let H be the subset of vectors in $\mathbb{R}^3$ consisting of those vectors whose first element is the sum of the second and third elements. Is H a subspace?
SOLUTION: One way of showing that a subset is a subspace is to show that the subspace can be represented by the span of some set of vectors. In this case,

$$\begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Because H is the span of the given vectors, it is a subspace.

(b) Explain why the image of a linear transformation $T : V \to W$ is a subspace of W

SOLUTION: Maybe "Prove" would have been better than "Explain", since we want to go through the three parts:

i. $0 \in T(V)$ since $0 \in V$ and $T(0) = 0$.

ii. Let u, v be in $T(V)$. Then there is an x, y in V so that $T(x) = u$ and $T(y) = v$. Since V is a subspace, $x + y \in V$, and therefore $T(x+y) = T(x)+T(y) = u+v$ so that $u + v \in T(V)$.

iii. Let $u \in T(V)$. Show that $cu \in T(V)$ for all scalars c. If $u \in T(V)$, there is an x in V so that $T(x) = u$. Since V is a subspace, $cu \in V$, and $T(cu) \in T(V)$. By linearity, this means $cT(u) \in T(V)$.

(OK, that probably should not have been in the short answer section)

(c) Is the following matrix diagonalizable? Explain. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$

SOLUTION: Yes. The eigenvalues are all distinct, so the corresponding eigenvectors are linearly independent.

(d) If the column space of an 8×4 matrix A is 3 dimensional, give the dimensions of the other three fundamental subspaces. Given these numbers, is it possible that the mapping $\mathbf{x} \to A\mathbf{x}$ is one to one? onto?

SOLUTION: If the column space is 3-d, so is the row space. Therefore the null space (as a subspace of $\mathbb{R}^4$) is 1 dimensional and the null space of A^T is 5 dimensional.

Since the null space has more than the zero vector, $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions, so the matrix mapping will not be 1-1. Since the column space is a three dimensional subspace of $\mathbb{R}^8$, the mapping cannot be onto.

6. True or False, and give a short reason:

(a) If A is 3×3, then $\det(5A) = 5\det(A)$.

FALSE. The determinant would be $5^3\det(A)$.

(b) If A, B are $n \times n$ with $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.

FALSE. We can break up the determinant of the product, but not the sum.

(c) If A is $n \times n$ and $\det(A) = 2$, then $\det(A^3) = 6$.

FALSE. The determinant should be $2^3 = 8$.

(d) If B is produced by taking row 1 and A and adding 3 times row 3, then putting the result back in row 1, then $\det(B) = 3\det(A)$.

FALSE. Using that row operation, the determinant remains unchanged.

7. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, then

(a) Find the solution to $A\mathbf{x} = \mathbf{b}$ using Cramer's Rule.
SOLUTION:

$$x_1 = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{vmatrix}} = \frac{12}{5} \quad x_1 = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{vmatrix}} = \frac{5}{5} = 1 \quad x_1 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{vmatrix}} = -\frac{4}{5}$$

(b) Find **only** the $(1, 2)$ entry of A^{-1} by using the formula for the adjoint.
SOLUTION: The $(1, 2)$ entry of A^{-1} is found by considering the 2d column of A^{-1}, which is found by solving $A\mathbf{x} = \mathbf{e}_2$. By Cramer's Rule, the first entry of the solution is then:

$$\frac{\det(A_1(\mathrm{e}_2))}{\det(A)} = \frac{C_{21}}{5} = \frac{(-1)\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix}}{5} = \frac{3}{5}$$

8. Find a basis for the null space, row space and column space of A, if $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

The basis for the column space is the set containing the first and third columns of A. A basis for the row space is the set of vectors $[1, 1, 0, 0]^T, [0, 0, 1, 1]^T$. A basis for the null space of A is $[-1, 1, 0, 0]^T, [0, 0, -1, 1]^T$.

9. Find an orthonormal basis for $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ using Gram-Schmidt:

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

SOLUTION:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

10. Using your answer to 9, if $\mathbf{y} = (1, 0, 0, 0)$, then find a vector in $\hat{\mathbf{y}} \in W$ and a vector in $\mathbf{z} \in W^{\perp}$ so that $\mathbf{y}$ is the sum of the vector in W and $W^{\perp}$.

SOLUTION: Projecting $\mathbf{y}$ to W, we only need to compute the projection to $\mathbf{v}_3$ since $\mathbf{y}$ is orthogonal to the other two vectors:

$$\hat{\mathbf{y}} = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{z} = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

11. (Referring to the previous problem) What is the distance between $\mathbf{y}$ and W?

    SOLUTION: The distance is $\|\mathbf{z}\| = \sqrt{\frac{1}{4}} = \frac{1}{2}$

12. If $\mathbf{x}_1, \mathbf{x}_2$ are the two vectors in problem 9, find a numerical expression for the angle between them.

    SOLUTION:

$$\cos(\theta) = \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{\|\mathbf{x}_1\|\,\|\mathbf{x}_2\|} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

    If you didn't see this last computation ($\pi/3$), I wouldn't mark it wrong- You should have gotten θ as the inverse cosine, though!

13. Let $\mathbb{P}_n$ be the vector space of polynomials of degree n or less. Let W_1 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(0)\mathbf{p}(1) = 0$. Let W_2 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(2) = 0$. Which of the two is a subspace of $\mathbb{P}_n$?

    SOLUTION: First consider W_1. The zero polynomial is in W_1. The condition $p(0)p(1) = 0$ means that either $p(0)$ or $p(1)$ are zero, but not necessarily both- That might cause a problem. Let's try the sum $f(t) = p(t) + q(t)$. If we expand the product:

$$f(0)f(1) = (p(0) + q(0))(p(1) + q(1)) = p(0)p(1) + p(0)q(1) + p(1)q(0) + q(0)q(1)$$

    The middle terms will cause a problem. For example, how about $p(0) = 4$, $p(1) = 0$ and $q(0) = 0$, but $q(1) = 3$? They each satisfy the condition for W_1, but the sum will not. Therefore, W_1 is not a subspace of $\mathbb{P}_n$.

    For W_2, we will have a subspace (check the three conditions).

14. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

    SOLUTION: For matrix A, $\lambda = 3, 5$. Eigenvectors are $[1, 1]^T$ and $[2, 1]^T$, respectively.

    For matrix B, for $\lambda = 3+i$, an eigenvector is $[1, i]^T$. The other eigenvalue and eigenvector are the complex conjugates.

    For matrix C, expand along the 2d row. $\lambda = 2$ is a double eigenvalue with eigenvectors $[0, 1, 0]^T$ and $[1, 0, 1]^T$. The third eigenvalue is $\lambda = 0$ with eigenvector $[-1, 0, 1]^T$.

15. Define $T : P_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

(a) Find the image under T of $p(t) = 5 + 3t$.
SOLUTION: $[2, 5, 8]^T$

(b) Show that T is a linear transformation.
SOLUTION: We show it using the definition.

i. Show that $T(p + q) = T(p) + T(q)$:

$$T(p+q) = \begin{bmatrix} p(-1) + q(-1) \\ p(0) + q(0) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} = T(p) + T(q)$$

ii. Show that $T(cp) = cT(p)$ for all scalars c.

$$T(cp) = \begin{bmatrix} cp(-1) \\ cp(0) \\ cp(1) \end{bmatrix} = c \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} = cT(p)$$

(c) Find the kernel of T. Does your answer imply that T is $1 - 1$? Onto? (Review the meaning of these words: kernel, one-to-one, onto)
SOLUTION:
Since the kernel is the set of elements in the domain that map to zero, let's see what what the action of T is on an arbitrary polynomial. An arbitrary vector in P_2 is: $p(t) = at^2 + bt + c$, and:

$$T(at^2 + bt + c) = \begin{bmatrix} a - b + c \\ c \\ a + b + c \end{bmatrix}$$

For this to be the zero vector, $c = 0$. Then $a - b = 0$ and $a + b = 0$, so $a = 0, b = 0$. Therefore, the only vector mapped to zero is the zero vector.
Side Remark: Recall that for any linear function T, if we are solving $T(x) = y$, then the solution can be written as $x = x_p + x_h$, where x_p is the particular solution (it solves $T(x_p) = y$), and $T(x_h) = 0$ (we said x_h is the homogeneous part of the solution). So the equation $T(x) = y$ has at most one solution iff the kernel is only the zero vector (if T was realized as a matrix, we get our familiar setting).

Therefore, T is $1 - 1$. The mapping T will also be onto (see the next part).

(d) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for P_2. (This means that the matrix will act on the *coordinates* of p).
The columns of the matrix are the coordinate vectors of the images of $T(1)$, $T(t)$ and $T(t^2)$, which are the following, since we're using the standard basis in $\mathbb{R}^3$:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

From this we see that the matrix is invertible, so the mapping is both 1-1 and onto (as a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$).

16. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ so that $\|\mathbf{v}\| = 1$, and let $Q = I - 2\mathbf{v}\mathbf{v}^T$. Show (by direct computation) that $Q^2 = I$.

    SOLUTION: This problem is to practice matrix algebra:

$$Q^2 = (I - 2\mathbf{v}\mathbf{v}^T)(I - 2\mathbf{v}\mathbf{v}^T) = I^2 - 2I\mathbf{v}\mathbf{v}^T - 2\mathbf{v}\mathbf{v}^T I + 4\mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T = I - 4\mathbf{v}\mathbf{v}^T + 4\mathbf{v}(1)\mathbf{v}^T = I$$

17. Let A be $m \times n$ and suppose there is a matrix C so that $AC = I_m$. Show that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$. Hint: Consider $AC\mathbf{b}$.

    SOLUTION: Using the hint, we see that $AC\mathbf{b} = \mathbf{b}$. Therefore, given an arbitrary vector $\mathbf{b}$, the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = C\mathbf{b}$.

18. If B has linearly dependent columns, show that AB has linearly dependent columns. Hint: Consider the null space.

    SOLUTION: If B has linearly dependent columns, then the equation $B\mathbf{x} = \mathbf{0}$ has non-trivial solutions. Therefore, the equation $AB\mathbf{x} = \mathbf{0}$ has (the same) non-trivial solutions, and the columns of AB must be linearly dependent.

19. If λ is an eigenvalue of A, then show that it is an eigenvalue of A^T.

    SOLUTION: Use the properties of determinants. Given

$$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I^T| = |A^T - \lambda I|$$

    the solutions to $|A - \lambda I| = 0$ and $|A^T - \lambda I| = 0$ are exactly the same.

20. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelogram with vertices at $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{u} + v$. Compute the area of S.

    SOLUTION: The area of the parallelogram formed by two vectors in $\mathbb{R}^2$ is the determinant of the matrix whose columns are those vectors. In this case, that would be 4.

21. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

    If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

    SOLUTION: This question reviews the relationship between the determinant and row operations. The determinant of B is 5. The determinant of C is 10. The determinant of BC is 50.

22. Let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix} \right\}$, and $\mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}$. Write down the matrices that take $[x]_C$ to $[x]_B$ and from $[x]_B$ to $[x]_C$.

    SOLUTION: Since $\mathbf{x} = P_B[\mathbf{x}]_B$ and $\mathbf{x} = P_C[\mathbf{x}]_C$, then

$$[\mathbf{x}]_C = P_C^{-1} P_B [\mathbf{x}]_B \qquad [\mathbf{x}]_B = P_B^{-1} P_C [\mathbf{x}]_C$$

Doing the calculations,

$$P_C^{-1}P_B = \begin{bmatrix} -1/2 & 5 \\ -1 & 5 \end{bmatrix} \qquad P_B^{-1}P_C = \begin{bmatrix} 2 & 2 \\ 2/5 & 1/5 \end{bmatrix}$$

23. Define an *isomorphism:* A one-to-one linear transformation between vector spaces (see p. 251)

24. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$

Find at least two $\mathcal{B}-$coordinate vectors for $\mathbf{x} = [1, 1]^T$.

SOLUTION: The null space of the matrix is spanned by $[5, -1, 1]^T$ (found by row reduction. The particular part of the solution is $[5, -2, 0]^T$. So we can find an infinite number of solutions.

25. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for vector space V. Explain why the $\mathcal{B}-$coordinate vectors of $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ are the columns of the $n \times n$ identity matrix:

SOLUTION: $[\mathbf{b}_1]_B = [1, 0, 0, \dots, 0]^T$ because $\mathbf{b}_1 = 1\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + 0\mathbf{b}_n$. A similar argument shows that $[\mathbf{b}_i]_B = \vec{e}_i$.

26. Find the volume of the parallelepiped formed by $\mathbf{0}$, $\mathbf{a}$,$\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}+\mathbf{b}$, $\mathbf{c}+\mathbf{b}$, $\mathbf{c}+\mathbf{a}$, and the sum of all three.

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The volume is the determinant. In this case, expand along the second row, and get 3.

27. Let $\mathbf{u} = (5, -6, 7)$. Let W be the set of all vectors orthogonal to $\mathbf{u}$.

SOLUTIONS:

(i) Geometrically, what is W? W is a plane in 3-d, through the origin, with normal vector $\mathbf{u}$.

(ii) Compute the projection of $\mathbf{x} = (1, 2, 3)$ onto W. To do this, we could find an spanning set for W, but you can also think about it this way-

$$\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$$

where $\mathbf{z}$ is the projection of $\mathbf{x}$ onto $W^\perp$, which is easy to compute. Then the desired vector would be:

$$\hat{\mathbf{x}} = \mathbf{x} - \mathbf{z}$$

Here we go:

$$\mathbf{z} = \frac{7}{55}\begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{7}{55}\begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$$

(You can leave it there- Sorry about the numbers!)

(iii) Write W as the span of some set (that is, find a basis for W).

SOLUTION: We're finding a basis for the null space of the matrix $[5 \quad -6 \quad 7]$, which can be written as:

$$\left\{ \begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 5 \end{bmatrix} \right\}$$

28. Suppose A is a 3×4 matrix, and any solution to $A\mathbf{x} = \mathbf{0}$ can be written as a linear combination:

$$\mathbf{x} = s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Remembering that A is 3×4, find the row reduced echelon form of A:

SOLUTION: We want to work backwards. Notice that $\mathbf{x}$ written above is set up to let x_3, x_4 be the two free variables. In that case, the corresponding system of equations would be 2. Since A has three rows, the third row is all zeros.

$$\begin{aligned} x_1 &= x_3 - 2x_4 \\ x_2 &= x_3 - x_4 \end{aligned} \quad \Rightarrow \quad \text{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find the dimensions of all four fundamental subspaces: Col(A), Row(A), Null(A), and Null(A^T).

SOLUTION: With 2 pivots, the column and row spaces each have 2 dimensions. The null space of A would then have 2 dimensions (the domain is $\mathbb{R}^4$), and the null space of A^T would have 1 dimension, since the codomain is $\mathbb{R}^3$.

(c) You have enough information to find bases for one or more of these subspaces- Find those bases.

SOLUTION: The basis for the null space is given in the problem. The basis for the row space is found by taking the two rows of the RREF of A above. We do not have enough information to find a basis for the column space (we would need the first two columns of the original matrix), so we also can't find a basis for the null space of A^T.

29. Suppose A is a 6×3 matrix and $A\mathbf{x} \neq \mathbf{0}$ if $\mathbf{x} \neq \mathbf{0}$.

(a) What can be said about the columns of A?

SOLUTION: The columns are all linearly independent, since there are no free variables. We can also say that the rank is 3.

(b) Show that $A^T A\mathbf{x} \neq \mathbf{0}$ (for $\mathbf{x} \neq \mathbf{0}$) by explaining this key step:

If $A^T A\mathbf{x} = \mathbf{0}$, then clearly $\mathbf{x}^T A^T A\mathbf{x} = 0$, and then (Why?) $A\mathbf{x} = \mathbf{0}$.

SOLUTION: $A\mathbf{x} = \mathbf{0}$ because $\mathbf{x}^T A^T A\mathbf{x} = \|A\mathbf{x}\|^2$, and the only vector with zero length is the zero vector.

(c) By the previous step, we know that $A^T A$ is invertible (Why?).

SOLUTION: The previous step showed that the null space of the 3×3 matrix $A^T A$ was the zero vector, so by the Invertible Matrix Theorem, $A^T A$ must be invertible.

30. Consider the system:

$$\begin{aligned} x + 2y - z &= 3 \\ x + 2y - z &= 2 \\ x + 2y - z &= -2 \end{aligned}$$

Clearly, the system is inconsistent. Find the least squares solution, and write the solution in (parametric) vector form.

SOLUTION: Using the normal equations, and row reducing since the matrix is rank 1:

$$[A^T A | A^T \mathbf{b}] = \left[\begin{array}{rrr|r} 3 & 6 & -3 & 3 \\ 6 & 12 & -6 & 6 \\ -3 & -6 & 3 & -3 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The least squares solution is the plane $x + 2y - z = 1$, which in parametric vector form is:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Side Remark: Was it necessary to form the normal equations? Yes! If you tried to do row reduction from the original equations, you'd get no solution- The RREF would have been

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$