Math 240, Second Exam REVIEW QUESTIONS

1. Find the inverse of the matrix A below:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

2. Suppose A, B and X are $n \times n$ matrices, with A, X, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B$$

First, explain why B is invertible, then solve the equation for X. If you need to invert a matrix, explain why it is invertible.

3. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$. Find A^{-1} using the formula, then solve $A\mathbf{x} = [3, 5]^T$.

4. Show that, if AB is invertible, then so is A (assume A, B are $n \times n$). Hint: If AB is invertible, then there is a matrix W so that $ABW = I$.

5. Let S be the parallelogram whose vertices are $(-1, 1), (0, 4), (1, 2)$ and $(2, 5)$. Use determinants to find the area of S.

6. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

7. Assume that A and B are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) State which vector space contains each of the four fundamental subspaces, and state the smallest number of vectors needed to span each space.

   (b) Find those vectors for Col(A):

   (c) Find those vectors for Row(A):

   (d) Find those vectors for Null(A):

8. Determine if the following sets are subspaces of V. Justify your answers.

- $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a \geq 0, \; b \geq 0, \; c \geq 0 \right\}, \qquad V = \mathbb{R}^3$

- $H = \left\{ \begin{bmatrix} a+3b \\ a-b \\ 2a+b \\ 4a \end{bmatrix}, \;\; a, b \text{ in } \mathbb{R} \right\}, \quad V = \mathbb{R}^4$
- $H = \{f : f'(x) = f(x)\}, V = C^1(-\infty, \infty)$
  (C^1 is the space of differentiable functions where the derivative is continuous).
- H is the set of vectors in $\mathbb{R}^3$ whose first entry is the sum of the second and third entries, $V = \mathbb{R}^3$.
- Let V be the space of continuous functions. Let H be the set of differentiable functions. Before you answer the question if H is a subspace, discuss whether or not H is a subset of V.

9. Prove that, if $T : V \mapsto W$ is a linear transformation between vector spaces V and W, then the range of T, which we denote as $T(V)$, is a subspace of W.

10. Let H, K be subspaces of vector space V. Define $H + K$ as the set below, and see if $H + K$ is a subspace (check all parts of the definition).

$$H + K = \{\mathbf{w} \,|\, \mathbf{w} = \mathbf{u} + \mathbf{v}, \text{ for some } \mathbf{u} \in H, \mathbf{v} \in K\}$$

11. Let A be an $n \times n$ matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement "A is invertible". Use the following concepts, one in each statement: (a) Null(A) (b) Span (c) Pivots (d) det(A)

12. Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

13. Use any part of the Invertible Matrix Theorem (IMT) to find the value(s) of s so that the matrix below is invertible.

$$\begin{bmatrix} s & 1 & 0 \\ 0 & s & 1 \\ 0 & 1 & s \end{bmatrix}$$

14. If A is given below (and the determinant of A is 14), find the $(3, 1)$ element of the inverse of A (only this element- do not fully invert the matrix!)

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

15. Let $T : V \to W$ be a $1-1$ and linear transformation on vector space V to vector space W. Show that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are linearly dependent vectors in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent vectors in V.

16. Use Cramer's Rule to solve the system:

$$\begin{array}{rrrl} 2x_1 & +x_2 & & = 7 \\ -3x_1 & & +x_3 & = -8 \\ & x_2 & +2x_3 & = -3 \end{array}$$

17. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$, and $\mathbf{w} = [2, 1]^T$. Is $\mathbf{w}$ in the column space of A? Is it in the null space of A?

18. Prove that the column space is a vector space using a very short proof, then prove it directly by showing the three conditions hold.

19. If A, B are 4×4 matrices with $\det(A) = 2$ and $\det(B) = -3$, what is the determinant of the following (if you can compute it): (a) $\det(AB)$, (b) $\det(A^{-1})$, (c) $\det(5B)$ (d) $\det(3A - 2B)$, (e) $\det(B^T)$

20. True or False, and give a short reason:

    (a) If $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.
    (b) Let A be $n \times n$. Then $\det(A^T A) \geq 0$.
    (c) If A^3 is the zero matrix, then $\det(A) = 0$.
    (d) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.
    (e) Row operations preserve the linear dependence relations among the rows of A.
    (f) If $BC = BD$, then $C = D$.
    (g) If $AB = I$, then A is invertible.
    (h) If A is 3×3 and $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then A is invertible.
    (i) If A is invertible, then $A^T A$ is invertible.

21. Let the matrix A and its RREF, R_A, be given as below:

$$A = \begin{bmatrix} 1 & 1 & 7 & 2 & 2 \\ 3 & 0 & 9 & 3 & 4 \\ -3 & 1 & -5 & -2 & 3 \\ 2 & 2 & 14 & 4 & 2 \end{bmatrix} \qquad R_A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, define Z and its RREF, R_Z, as:

$$Z = \begin{bmatrix} 4 & 5 & 3 & 4 \\ 5 & 6 & 5 & -3 \\ 10 & -3 & 9 & -106 \\ 4 & 10 & 2 & 44 \end{bmatrix} \qquad R_z = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the smallest number of columns for A that still spans the column space of A. Similarly, do the same for Z:

(b) Here is a row reduction using some columns of A and Z:

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 4 & 5 & 3 \\ 3 & 0 & 4 & 5 & 6 & 5 \\ -3 & 1 & 3 & 10 & -3 & 9 \\ 2 & 2 & 2 & 4 & 10 & 2 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Are the subspaces spanned by the columns of A and Z equal?

22. Find the determinant of the matrix A below:

$$A = \begin{bmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{bmatrix}$$

23. Let V, W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace Z of W, define set U as the preimage of Z in V:

$$U = \{x \in V \mid T(x) \in Z\}$$

Show that U is a subspace.

24. Consider $\mathbb{P}_n$, the set of all polynomials of degree n or less. Consider the set of polynomials in $\mathbb{P}_n$ where $p(0) = 0$. Show that this set is or is not a subspace.

25. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ by: $T(x_1, x_2, x_3) = (x_1, x_1 + 2x_2, x_1 + 2x_2 + 3x_3)$.

(a) Is T an invertible function? Explain.

(b) If T is invertible, find a matrix so that $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$