MATH 240, Second Exam Review Solutions

1. The inverse is computed by augmenting A with I, then performing row reduction until A becomes I (and I becomes A^{-1}).

$$A^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

2. Suppose A, B and X are $n \times n$ matrices, with A, X, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B$$

First, explain why B is invertible, then solve the equation for X. If you need to invert a matrix, explain why it is invertible.

SOLUTION: B is invertible because it is the product of invertible matrices- That is,

$$B = X(A - AX)^{-1}$$

We want to isolate X. We could start by getting rid of the other inverse by right-multiplication, then distribute B through. Get all X on one side:

$$B(A - AX) = X \quad \Rightarrow \quad BA - BAX = X \quad \Rightarrow \quad BA = X + BAX$$

Factor out X. Is $I + BA$ invertible?

$$BA = (I + BA)X$$

In the equation above, A, B, X are all invertible, so $I + BA$ must also be invertible. Therefore, one way to write X is:

$$(I + BA)^{-1}BA = X$$

This is only one way to isolate X- There are other valid expressions as well. What's important in this problem is your step-by-step justifications.

3. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$. Find A^{-1} using the formula, then solve $A\mathbf{x} = [3, 5]^T$.

SOLUTION: First, we see that

$$A^{-1} = \frac{1}{12 - 10} \begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}$$

and

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$$

4. Show that, if AB is invertible, then so is A (assume A, B are $n \times n$). Hint: If AB is invertible, then there is a matrix W so that $ABW = I$.

   SOLUTION: Using the hint, $ABW = I$, or more precisely, there is a matrix $D = BW$ so that $AD = I$. Then by the Invertible Matrix Theorem (p 112, part (k)), A is invertible.

5. Let S be the parallelogram whose vertices are $(-1, 1), (0, 4), (1, 2)$ and $(2, 5)$. Use determinants to find the area of S.

   SOLUTION:

   If you plot the points, you should see which vectors we can take for the determinant:

$$\mathbf{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   Using these as the columns of the matrix A, the area is: $|1 - 6| = 5$.

6. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

   If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

   SOLUTION:

   To get B from A, we take $2r_3 + r_1 \to r_1$ and $3r_3 + r_2 \to r_2$. These row operations do not change the determinant, so $\det(A) = \det(B) = 5$.

   To get the determinant of C, multiply the determinant of A by -2 (negative because of the row swap); we get -10.

   The determinant of BC is then $5 \cdot -10 = -50$

7. Assume that A and B are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) State which vector space contains each of the four subspaces, and state the smallest number of vectors needed to span each space:

   SOLUTION: Here are the spaces and dimensions:

   i. The row space of A is a subspace of $\mathbb{R}^5$ and we'll need 3 vectors (because of 3 pivot rows).

   ii. The column space of A is a subspace of $\mathbb{R}^4$ and we'll need 3 vectors (because of 3 pivot columns).

iii. The null space of A is a subspace of $\mathbb{R}^5$ and we'll need 2 vectors (because 2 free variables).

iv. The null space of A^T is a subspace of $\mathbb{R}^4$ and we'll need 1 vector (because there is one non-pivot row).

(b) Find a smallest spanning set for Col(A): We see from the RREF that the pivot columns are cols 1, 2, and 4. Be sure to use the columns from the original matrix A! The set is then:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix} \right\}$$

(c) Find a smallest spanning set for Row(A): We can use the reduced rows (written as columns) for the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(d) Find a smallest spanning set for Null(A): Solve $A\mathbf{x} = \mathbf{0}$:

$$\begin{array}{ll} x_1 & = -4x_3 + 3x_5 \\ x_2 & = 3x_3 - 5x_5 \\ x_3 & = x_3 \\ x_4 & = \qquad\quad 4x_5 \\ x_5 & = \qquad\quad\ \ x_5 \end{array} \Rightarrow \left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

HINT: Double check that your vectors are in the right spaces- For example, your vectors from the null space are in $\mathbb{R}^5$, columns from the column space are in $\mathbb{R}^4$, etc.

8. Determine if the following sets are subspaces of V. Justify your answers.

- $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a \geq 0, \quad b \geq 0, \quad c \geq 0 \right\}, \qquad V = \mathbb{R}^3$

  SOLUTION: This set is not closed under scalar multiplication (If r is the scalar, then $r = -1$ would take the vector out of H).

- $H = \left\{ \begin{bmatrix} a + 3b \\ a - b \\ 2a + b \\ 4a \end{bmatrix}, \quad a, b \text{ in } \mathbb{R} \right\}, \quad V = \mathbb{R}^4$

  SOLUTION: This is the span of $[1, 1, 2, 4]^T$ and $[3, -1, 1, 0]^T$, so H is a subspace.

- $H = \{f : f'(x) = f(x)\}, V = C^1[\mathbb{R}]$
  (C^1 is the space of differentiable functions where the derivative is continuous).
  SOLUTION: We could show this directly:
  - The zero function is in H since the derivative of 0 is 0.
  - If u, v are in H, then $u' = u$ and $v' = v$. Therefore, $(u+v)' = u' + v' = u + v$, so $u + v$ is in H.
  - If $u \in H$, then $u' = u$. Therefore, $(cu)' = cu' = cu$, so cu is in H.

  Side Remark, for students that have had differential equations- In that class, we showed that H is spanned by one function, $y = \mathrm{e}^x$.
- H is the set of vectors in $\mathbb{R}^3$ whose first entry is the sum of the second and third entries, $V = \mathbb{R}^3$.
  SOLUTION: Rewriting H algebraically, it is the set of vectors in $\mathbb{R}^3$ so that

$$\begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

  Since H is the span of two vectors, it is a subspace.
- Let V be the space of continuous functions. Let H be the set of differentiable functions. Before you answer the question if H is a subspace, discuss whether or not H is a subset of V . The set of continuous functions, V, contains H since every differentiable function is "automatically" continuous (on the same domain). This is a "proper" subset since there are continuous functions that are not differentiable (like the absolute value function).
  Proving that H is a subspace:
  - Is $\vec{0} \in H$. Yes, since $f(x) = 0$ is differentiable (and $f'(x) = 0$).
  - Let f, g be in H. Therefore, f and g are differentiable, and:

$$(f + g)'(x) = f'(x) + g'(x)$$

    so the sum of differentiable functions is differentiable.
  - Let f be in H. Show that $cf \in H$: From calculus, we know that if f is differentiable, then

$$(cf)'(x) = cf'(x)$$

    so that cf is differentiable (and so is in H for all c).

9. Prove that, if $T : V \mapsto W$ is a linear transformation between vector spaces V and W, then the range of T, which we denote as $T(V)$, is a subspace of W.

   SOLUTION: Show that the three parts to the definition hold true-

   - Since $\mathbf{0} \in V$ (V is a subspace) and $T(\mathbf{0}) = \mathbf{0}$, then $\mathbf{0} \in T(V)$.

- Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be in $T(V)$. Show that the sum is in $T(V)$.

  Since $\mathbf{w}_1, \mathbf{w}_2 \in T(V)$, there are vectors, $\mathbf{v}_1, \mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Now by linearity, we have:

$$\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2)$$

  Therefore, $\mathbf{w}_1 + \mathbf{w}_2$ is the image of $\mathbf{v}_1 + \mathbf{v}_2$ which is in V (because V is a subspace). Therefore, $\mathbf{w}_1 + \mathbf{w}_2$ is in $T(V)$.

- Let $\mathbf{w} \in T(V)$, and show $c\mathbf{w} \in T(V)$.

  Since $\mathbf{w} \in T(V)$, there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. We also know that $c\mathbf{v} \in V$ because V is a subspace. By the linearity of T,

$$c\mathbf{w} = cT(\mathbf{v}) = T(c\mathbf{v})$$

  and so $c\mathbf{w} \in T(V)$.

10. Let H, K be subspaces of vector space V. Define $H + K$ as the set below:

$$H + K = \{\mathbf{w} \mid \mathbf{w} = \mathbf{u} + \mathbf{v}, \text{ for some } \mathbf{u} \in H, \mathbf{v} \in K\}$$

SOLUTION:

- Since $\mathbf{0} \in H$ and K, and $\mathbf{0} + \mathbf{0} = \mathbf{0}$, then $\mathbf{0} \in H + K$
- Let $\mathbf{w}_1, \mathbf{w}_2 \in H + K$. Show that the sum is in $H + K$:

  Since $\mathbf{w}_1, \mathbf{w}_2 \in H + K$, there are vectors, $\mathbf{u}_1, \mathbf{u}_2$ in H and vectors, $\mathbf{v}_1, \mathbf{v}_2$ in K such that

$$\begin{array}{ll} \mathbf{w}_1 & = \mathbf{u}_1 + \mathbf{v}_1 \\ \mathbf{w}_2 & = \mathbf{u}_2 + \mathbf{v}_2 \end{array} \Rightarrow \mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$$

  with $\mathbf{u}_1 + \mathbf{u}_2 \in H$ and $\mathbf{v}_1 + \mathbf{v}_2 \in K$ because H, K are subspaces.

- Let $\mathbf{w} \in H + K$. Show that $c\mathbf{w} \in H + K$:

  If $\mathbf{w} \in H + K$, then there are vectors $\mathbf{u} \in H$ and $\mathbf{v} \in K$ so that $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Therefore,

$$c\mathbf{w} = c\mathbf{u} + c\mathbf{v}$$

  and since H and K are subspaces, $c\mathbf{u} \in H$ and $c\mathbf{v} \in K$. Therefore, $c\mathbf{w}$ is a sum of something in H and something in K, so it is in $H + K$.

11. Let A be an $n \times n$ matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement "A is invertible". Use the following concepts, one in each statement: (a) Null(A) (b) span (c) Pivots (d) det

SOLUTION:

(a) Null(A) is only the zero vector (b) cols of A span $\mathbb{R}^n$
(c) Number of pivots is n. (d) The det is not zero.

12. Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

SOLUTION:

No. If we have 10 equations in 12 variables, we must have at least two free variables, and so the null space of the corresponding matrix is the span of at least two linearly independent (nonzero) vectors.

13. Use any part of the IMT to find the value(s) of s so that the matrix is invertible.

SOLUTION: Probably the most straightforward is to find that the determinant is $s(s^2-1)$. Therefore, as long as $s \neq 0, \pm 1$, then the determinant is non-zero and the matrix is invertible.

14. If A is given below (and the det of A is 14), find the $(3,1)$ element of A^{-1}.

SOLUTION: We use the adjugate (or adjoint) formula. As a hint, you might recall that we got this from Cramer's rule- The $(3,1)$ element is in the first column of A^{-1}, so from using Cramer's rule (solving $A\mathbf{x} = \mathbf{e}_1$), we want only the third value of $\mathbf{x}$, and that is from replacing the third column of A with $\mathbf{e}_1$:

$$\frac{\det(A_3(\mathbf{e}_1))}{\det(A)} = \frac{(-1)^{3+1}\det(A_{13})}{14} = \frac{C_{13}}{14} = \frac{5}{14}$$

15. Let $T : V \to W$ be a $1-1$ and linear transformation on vector space V to vector space W. Show that if $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are linearly dependent vectors in W, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent vectors in V.

SOLUTION:

If $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are linearly dependent, then there is a nontrivial solution to

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = 0$$

Since T is linear, we can write this equation as:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(0)$$

If T is $1-1$, we can say that $T(a) = T(b)$ implies that $a = b$. In this case, we then can say that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

So that we again get a non-trivial solution, and these vectors are linearly dependent.

16. Use Cramer's Rule to solve the system:

$$\begin{aligned} 2x_1 + x_2 &= 7 \\ -3x_1 + x_3 &= -8 \\ x_2 + 2x_3 &= -3 \end{aligned}$$

SOLUTION: We'll need some determinants:

$$\begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4 \qquad \begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 6 \qquad \begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix} = 16 \qquad \begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix} = -14$$

So the solution is:

$$x_1 = \frac{6}{4} = \frac{3}{2} \qquad x_2 = \frac{16}{4} = 4 \qquad x_3 = \frac{-14}{4} = \frac{-7}{2}$$

17. Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$, and $\mathbf{w} = [2, 1]^T$. Is $\mathbf{w}$ in the column space of A? Is it in the null space of A?

    SOLUTION: The first question is asking if $\mathbf{w}$ is in the span of the columns of A. By inspection, the weights are $-1/3$ and 0, so yes.

    The second question is asking if $A\mathbf{w} = \mathbf{0}$. In this case, it is true as well.

    Side Note: It was only a coincidence that this vector was in both the column space as well as the null space. If A was $m \times n$, then the column space would be in $\mathbb{R}^m$ and the null space would be in $\mathbb{R}^n$, so these would definitely be different.

18. Prove that the column space is a vector space using a very short proof, then prove it directly by showing the three conditions hold.

    SOLUTION:

    For the short proof, we know that any spanning set is automatically a subspace. The column space is the span of the columns of A.

    Now for the longer proof using the definition:

    Recall that a vector $\mathbf{b}$ is in the column space of A if and only if $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$.

    (a) Since $A\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in \text{Col}(A)$.

    (b) Let $\mathbf{b}_1, \mathbf{b}_2 \in \text{Col}(A)$. Then there exists $\mathbf{x}_1, \mathbf{x}_2$ so that

    $$A(\mathbf{x}_1) = \mathbf{b}_1 \qquad A(\mathbf{x}_2) = \mathbf{b}_2 \text{ therefore } \mathbf{b}_1 + \mathbf{b}_2 = A(\mathbf{x}_1) + A(\mathbf{x}_2) = A(\mathbf{x}_1 + \mathbf{x}_2)$$

    This shows that $\mathbf{b}_1 + \mathbf{b}_2$ is in the column space of A.

    (c) Let $\mathbf{b} \in \text{Col}(A)$. Then $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{x}$ in the domain, and

    $$c\mathbf{b} = cA\mathbf{x} = A(c\mathbf{x})$$

    Therefore, $c\mathbf{b}$ is in the column space of A for all constants c.

19. If A, B are 4×4 matrices with $\det(A) = 2$ and $\det(B) = -3$, what is the determinant of the following (if you can compute it):

SOLUTIONS:

(a) $\det(AB) = -6$, (b) $\det(A^{-1}) = 1/2$, (c) $\det(5B) = 5^4(-3)$, (d) $\det(3A - 2B)$ is unknown with what is given, (e) $\det(B^T) = -3$

20. True or False, and give a short reason:

(a) If $\det(A) = 2$ and $\det(B) = 3$, then $\det(A + B) = 5$.

SOLUTION: False. $\det(A + B) \neq \det(A) + \det(B)$.

(b) Let A be $n \times n$. Then $\det(A^T A) \geq 0$.

SOLUTION: True: $\det(A^T A) = \det(A^T)\det(A) = (\det(A))^2 \geq 0$.

(c) If A^3 is the zero matrix, then $\det(A) = 0$.

SOLUTION: True- If $A^3 = 0$, then $0 = \det(A^3) = (\det(A))^3$, so $\det(A) = 0$.

(d) $\mathbb{R}^2$ is a two dimensional subspace of $\mathbb{R}^3$.

SOLUTION: False. $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, they are completely different vector spaces. However, a two dimensional subspace in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

(e) Row operations preserve the linear dependence relations among the rows of A.

SOLUTION: False. Swapping rows will especially change the relations- However, row operations do NOT change the linear dependence relations among the **columns** of A.

(f) If $BC = BD$, then $C = D$

SOLUTION: False in general. If B is invertible, then the statement is true.

(g) If $AB = I$, then A is invertible.

SOLUTION: False in general- A may not even be square. However, if A, B are $n \times n$, then the statement is true (by the IMT).

(h) If A is 3×3 and $A\mathbf{x} = (1, 0, 0)$ has a unique solution, then A is invertible.

SOLUTION: This is true. If A is 3×3 and $A\mathbf{x} = (1, 0, 0)$ has a unique solution, then there are no free variables, so every column is a pivot column. By the IMT then, A is invertible.

21. Let the matrix A and its RREF, R_A, be given as below:

$$A = \begin{bmatrix} 1 & 1 & 7 & 2 & 2 \\ 3 & 0 & 9 & 3 & 4 \\ -3 & 1 & -5 & -2 & 3 \\ 2 & 2 & 14 & 4 & 2 \end{bmatrix} \qquad R_A = \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, define Z and its RREF, R_Z, as:

$$Z = \begin{bmatrix} 4 & 5 & 3 & 4 \\ 5 & 6 & 5 & -3 \\ 10 & -3 & 9 & -106 \\ 4 & 10 & 2 & 44 \end{bmatrix} \qquad R_z = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the smallest number of columns for A that still spans the column space of A. Similarly, do the same for Z:

SOLUTION: The RREF of A is telling us that columns 1, 2 and 5 of the original matrix are linearly independent, and the other values are telling us that columns 3, 4 are combinations of columns 1 and 2. Therefore, we only need 1, 2, and 5 from the original matrix A.

Similarly, for Z, the RREF of Z says that the first three columns are linearly independent, and the fourth column is a linear combination of the first three. Therefore, we only need the first three columns of Z.

(b) Here is a row reduction using some columns of A and Z:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 5 & 3 \\ 3 & 0 & 4 & 5 & 6 & 5 \\ -3 & 1 & 3 & 10 & -3 & 9 \\ 2 & 2 & 2 & 4 & 10 & 2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Are the subspaces spanned by the columns of A and Z equal?

SOLUTION: From the RREF given above, we see that the first three columns of Z can be written as a combination of columns 1, 2, and 5 of matrix A. Therefore, the spanning set will be the same.

22. Find the determinant of the matrix A below:

$$A = \begin{bmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{bmatrix}$$

SOLUTION: Expand in terms of the second row, so the determinant is the determinant of the smaller and smaller matrices:

$$\begin{vmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 8 & 8 & 5 \\ 6 & 8 & 8 & 7 \\ 0 & 8 & 3 & 0 \\ 0 & 2 & 0 & 0 \end{vmatrix} = 2 \begin{vmatrix} 4 & 8 & 5 \\ 6 & 8 & 7 \\ 0 & 3 & 0 \end{vmatrix} = 2 \cdot (-3) \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = 12$$

23. Let V, W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace Z of W, define set U as the preimage of Z in V:

$$U = \{x \in V \mid T(x) \in Z\}$$

Show that U is a subspace.

SOLUTION:

(a) Is $\vec{0} \in U$? Since Z is a subspace, we know $\vec{0} \in Z$, and since T is linear, then $T(\vec{0}) = \vec{0}$. Therefore, $\vec{0} \in U$.

(b) Is U closed under addition? Let $\mathbf{x}_1, \mathbf{x}_2$ be two vectors in U. By definition of U,

$$T(\mathbf{x}_1) = \mathbf{z}_1 \in Z, \qquad T(\mathbf{x}_2) = \mathbf{z}_2 \in Z$$

And since Z is a subspace, we know that $z_1 + z_2 \in Z$, and because T is linear:

$$z_1 + z_2 = T(\mathbf{x}_1) + T(\mathbf{x}_2) = T(\mathbf{x}_1 + \mathbf{x}_2)$$

so therefore, $\mathbf{x}_1 + \mathbf{x}_2 \in U$.
A similar argument holds for the last bit:

(c) Is U closed under scalar multiplication? Let $\mathbf{x} \in U$. Then

$$T(\mathbf{x}) = \mathbf{z} \in Z \quad \Rightarrow \quad cT(\mathbf{x}) = c\mathbf{z}$$

And, since Z is a subspace, $c\mathbf{z} \in Z$, and from the linearity of T,

$$c\mathbf{z} = cT(\mathbf{x}) = T(c\mathbf{x})$$

24. Consider $\mathbb{P}_n$, the set of all polynomials of degree n or less. Consider the set of polynomials in $\mathbb{P}_n$ where $p(0) = 0$. Show that this set is or is not a subspace.

SOLUTION: We'll call the set of polynomials where $p(0) = 0$ the subset H.

(a) Show that $0 \in H$. If $p(t) = 0$, then $p(0) = 0$, so $0 \in H$.

(b) Let $p_1, p_2 \in H$. Then: $(p_1 + p_2)(t) = p_1(t) + p_2(t)$, and the sum evaluated at $t = 0$ will be:

$$(p_1 + p_2)(0) = p_1(0) + p_2(0) = 0 + 0 = 0$$

Therefore, $p_1 + p_2 \in H$.

(c) Let $p \in H$. Then $p(0) = 0$, and $cp(0) = 0$, so that $cp \in H$ as well.

25. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ by: $T(x_1, x_2, x_3) = (x_1, x_1 + 2x_2, x_1 + 2x_2 + 3x_3)$.

(a) Is T an invertible function? Explain.

We see that we can build a matrix A so that $T(x) = Ax$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} \quad \Rightarrow \quad \det(A) \neq 0$$

Since the determinant is not zero, A is an invertible matrix, so T is an invertible function.

(b) If T is invertible, find a matrix so that $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$

SOLUTION: We should find that

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$