Example Questions, Exam 3, Math 240

1. Assume that A and B are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) State which vector space contains each of the four subspaces, and state the dimension of each of the four subspaces:
   (b) Find a basis for Col(A):
   (c) Find a basis for Row(A):
   (d) Find a basis for Null(A):

2. Suppose that T is a one-to-one linear transformation between vector spaces V and W. Show that, if a set of images, $\{T(\mathbf{v}_1, \ldots, T(\mathbf{v}_p)\}$ are linearly dependent, then so is the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$

3. In class we said that $\mathbb{P}_2$ is **isomorphic** to $\mathbb{R}^3$. (i) What is an isomorphism in general? (ii) What is the isomorphism in this particular case?

4. The set $\mathcal{B} = \left\{1 + t^2, t + t^2, 1 + 2t + t^2\right\}$ is a basis for $\mathbf{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1{+}4t{+}7t^2$ relative to $\mathcal{B}$.

5. Let A be an $n \times n$ matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement "A is invertible". Use the following concepts, one in each statement: (a) Null(A) (b) Basis (c) Rank (d) det(A) (e) Eigenvalue

6. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution, with two free variables. Is it possible to change some constants on the equation's right sides to make the new system inconsistent? Explain.

7. Is is possible for a nonhomogeneous system of seven equations in six unknowns to have a unique solution for some right hand side of constants? Is it possible for such a system to have a unique solution for every right hand side? Explain.

8. Short Answer:

   (a) T/F and give a short reason: If there is a set $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ that spans V, then the $dim(V) \leq p$.
   (b) Write the complex number in $a + ib$ form: $\frac{1-3i}{2+i}$.
   (c) Write the complex number in polar form, $re^{i\theta}$: $-1 + i$
   (d) If A is similar to B, then prove that A^2 is similar to B^2.

9. Show that $\mathcal{B} = \left\{1, 2t, -2 + 4t^2\right\}$ is a basis for P_2. Be explicit about your reasoning!

10. Assume the mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by the following is linear.

$$T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$$

    Find the matrix representation of T relative to the standard basis $\mathcal{B} = \left\{1, t, t^2\right\}$

11. If each row of the matrix A sums to the same number r, and A is $n \times n$, then what must one eigenvalue of A be, and what eigenvector? (Hint: Is there a vector $\mathbf{v}$ so that $A\mathbf{v}$ is a vector of row sums?)

12. Show that the eigenvalues of A and A^T are the same.

13. Find the eigenvalues and bases for the eigenspaces for each matrix below. If the matrix is diagonalizable in either PDP^{-1} or PCP^{-1} form, given P and D or C. Otherwise, state that as well.

(a) $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$. (c) $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$. (d) $A = \begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix}$.

14. If A is similar to B, show that they have the same eigenvalues. Do they also have the same eigenvectors?

15. Prove that, if $n \times n$ matrix A is not invertible, then $\lambda = 0$ is an eigenvalue.

16. Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.

17. Let $\mathcal{B}, \mathcal{C}$ be two sets of p vectors that represent bases for vector space V. Explain how we find $P_{\mathcal{C} \leftarrow \mathcal{B}}$. Hint: You might start with a vector $\mathbf{x} \in V$ and expand it relative to $\mathcal{B}$.

18. True or False? If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

    (a) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
    (b) If A is invertible, then A is diagonalizable.
    (c) A is diagonalizable if $A = PDP^{-1}$ for some matrix D and invertible matrix P.
    (d) A is diagonalizable if A has n eigenvalues, counting multiplicities.
    (e) If A, B are similar, they have the same rank.
    (f) If A, B are row equivalent, then they have the same row space.
    (g) If $\mathbf{u}, \mathbf{v}$ are eigenvectors of A, so is $\mathbf{u} + \mathbf{v}$.
    (h) The rank of a matrix is equal to the number of nonzero rows.

19. Show that the rank of AB is less than or equal to the rank of A. Hint: Think about the columns of AB.

20. Suppose $A = PDP^{-1}$ with a suitable 2×2 matrix P and $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

    (a) If $B = 3I - 2A + A^2$, show that B is diagonalizable by finding an appropriate factorization of B.
    (b) From your previous answer, if λ is an eigenvalue of A, then what would an eigenvalue of $A^2+bA+cI$ be?

21. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ are two sets of bases for the same vector space V. How do we find the matrix that represents a change of coordinates from $\mathcal{B}$ to $\mathcal{C}$? (Hint: Start with a vector in V written in terms of $\mathcal{B}$.)

22. In $\mathbb{P}_2$, let $\mathcal{B} = \left\{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\right\}$. Let $\mathcal{C} = \left\{1, t, t^2\right\}$.

    (a) Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.
    (b) Find the coordinates of t^2 with respect to the basis $\mathcal{B}$.

23. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis for some vector space V. Let $T : \mathbb{R}^3 \to V$ be a linear transformation:

$$T(x_1, x_2, x_3) = (2x_3 - x_2)\mathbf{c}_1 - (2x_2)\mathbf{c}_2 + (x_1 + 3x_3)\mathbf{c}_3$$

    Find the matrix for T relative to $\mathcal{E}$ and $\mathcal{C}$.

24. Find $T(a_0 + a_1 t + a_2 t^2)$ if T is the linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$ whose matrix relative to $\mathcal{B} = \left\{1, t, t^2\right\}$ is given by:

$$[T]_B = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

25. Suppose in Matlab, I have the matrix A below, and the program gave the following matrices for V and D. Find P, C so that $A = PCP^{-1}$.

$$A = \begin{bmatrix} 3 & 2 & 3 & 1 \\ 1 & 3 & 3 & 3 \\ 3 & 1 & 3 & 3 \\ 3 & 2 & 2 & 3 \end{bmatrix}$$

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>> [V,D]=eig(A)
V =
-0.4  -0.1 + 0.4i  -0.1 - 0.4i  -0.3
-0.5  -0.5 + 0.0i  -0.5 - 0.0i   0.7
-0.5   0.1 - 0.4i   0.1 + 0.4i  -0.4
-0.5   0.4 + 0.0i   0.4 - 0.0i   0.2

D =
9.7   0            0            0
0     0.2 + 1.3i   0            0
0     0            0.2 - 1.3i   0
0     0            0            1.7
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26. If we think about a block form of a matrix, if we have

$$G = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$$

where A, B are square (not necessarily the same size), then $\det(G) = (\det A)(\det(B))$. Use this to help find the eigenvalues of the matrices below:

$$G_1 = \begin{bmatrix} 3 & -2 & 8 \\ 0 & 3 & -2 \\ 0 & 2 & 3 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 5 & -6 & -7 \\ 2 & 4 & 5 & 2 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$