Review Solutions, Exam 3, Math 244

1. Assume that A and B are row equivalent, where:

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) State which vector space contains each of the four subspaces, and state the dimension of each of the four subspaces:

   SOLUTION: Here are the spaces and dimensions:

      i. The row space of A is a subspace of $\mathbb{R}^5$ and has dimension 3 (this is the number of pivot rows).
      ii. The column space of A is a subspace of $\mathbb{R}^4$ and has dimension 3 (this is also known as the rank of A and is the number of pivot columns).
      iii. The null space of A is a subspace of $\mathbb{R}^5$ and has dimension 2 (the number of free variables).
      iv. The null space of A^T is a subspace of $\mathbb{R}^4$ and has dimension 1 (because the dimension of the column space is 3, and they should add to 4).

   (b) Find a basis for Col(A): We see from the RREF that the pivot columns are cols 1, 2, and 4. Be sure to use the columns from the original matrix A! The basis is the set

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix} \right\}$$

   (c) Find a basis for Row(A): We can use the reduced rows (written as columns) for the basis:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right\}$$

   (d) Find a basis for Null(A): Solve $A\mathbf{x} = \mathbf{0}$, and the vectors are the basis vectors:

$$\begin{array}{ll} x_1 & = -4x_3 + 3x_5 \\ x_2 & = 3x_3 - 5x_5 \\ x_3 & = x_3 \\ x_4 & = \qquad 4x_5 \\ x_5 & = \qquad\quad x_5 \end{array} \quad \Rightarrow \left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

2. Suppose that T is a one-to-one linear transformation between vector spaces V and W. Show that, if a set of images, $\{T(\mathbf{v}_1, \ldots, T(\mathbf{v}_p)\}$ are linearly dependent, then so is the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$

   SOLUTION: If the set of images are linearly dependent, there is a non-trivial solution to:

$$c_1 T(\mathbf{v}_1 + \cdots + c_p T(\mathbf{v}_p) = 0$$

   Because T is a linear transformation, the equation can be rewritten as:

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = 0$$

   Because T is one-to-one, the sum of vectors must be zero:

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

   The $c's$ are not all zero, so this is a non-trivial solution. Therefore, the $\mathbf{v}'s$ must be linearly dependent.

3. In class we said that $\mathbb{P}_2$ is **isomorphic** to $\mathbb{R}^3$.

   (i) What is an isomorphism in general?

   SOLUTION: An isomorphism is a linear transformation that is one-to-one and onto.

   (ii) What is the isomorphism in this particular case?

   SOLUTION: The isomorphism is the coordinate mapping, in this case, if $\mathcal{B}$ is a basis for $\mathbb{P}_2$, then

$$\mathbf{p}(t) \in \mathbb{P}_2 \to [\mathbf{p}(t)]_B \in \mathbb{R}^3$$

4. The set $\mathcal{B} = \left\{1+t^2, t+t^2, 1+2t+t^2\right\}$ is a basis for $\mathbf{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1+4t+7t^2$ relative to $\mathcal{B}$.

   SOLUTION: We want to find the coordinates of $1+4t+7t^2$ with respect to the given basis. We can either work in $\mathbb{P}_2$ first, then go to the matrix:

$$1 + 4t + 7t^2 = c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2)$$

   which has the same solution as the matrix-vector equation:

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

   Therefore, $[\mathbf{p}(t)]_B = (2, 6, -1)$

5. Let A be an $n \times n$ matrix. Write statements from the Invertible Matrix Theorem that are each equivalent to the statement "A is invertible". Use the following concepts, one in each statement:

   (a) Null(A) is only the zero vector.

   (b) Basis: The columns form a basis for $\mathbb{R}^n$.

   (c) Rank: The rank of A is n

   (d) det(A): The determinant is not zero.

   (e) Eigenvalue: No eigenvalue is zero.

6. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution, with two free variables. Is it possible to change some constants on the equation's right sides to make the new system inconsistent? Explain.

   SOLUTION: Think about the four fundamental subspaces. In this case, the matrix A is 6×8, so the column space is in $\mathbb{R}^6$ and the row space is in $\mathbb{R}^8$.

   If a nonhomogeneous system has a solution with two free variables, then the null space is a two dimensional subspace of $\mathbb{R}^8$.

   The question asks if the system is "onto". To answer this, we need to know the dimension of the column space. From what we're given, if the null space is two dimensional, the row space is then 6 dimensional, which must be the dimension of the column space, which is in $\mathbb{R}^6$. Therefore, the columns fill $\mathbb{R}^6$, and so it is not possible to make the system inconsistent.

7. Is is possible for a nonhomogeneous system of seven equations in six unknowns to have a unique solution for some right hand side of constants? Is it possible for such a system to have a unique solution for every right hand side? Explain.

   SOLUTION: Same idea as before. In this case, the matrix is 7×6. We note that it is possible that every column is a pivot column, so IF the system is consistent, it is possible for the system to have a unique solution.

   Is it possible to have a unique solution for EVERY right hand side? No. Since we'll have a row of zeros in the RREF of A, the system will not be consistent for some right hand sides.

8. Short Answer:

   (a) T/F and give a short reason: If there is a set $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ that spans V, then the $dim(V) \leq p$.
   TRUE. If the set is linearly dependent, we can remove the linearly dependent vectors until we get a linearly independent set. Therefore, the dimension is less than p. If the set were linearly independent to start with, the dimension is p.

   (b) Write the complex number in $a + ib$ form: $\frac{1-3i}{2+i}$.
   SOLUTION: Multiply top and bottom by $2 - i$:
   $$\frac{(1-3i)(2-i)}{2^2+1^2} = \frac{2-i-6i+3i^2}{5} = -\frac{1}{5} - \frac{7}{5}i$$

   (c) Write the complex number in polar form, $re^{i\theta}$: $-1 + i$
   SOLUTION: Think of the point in the plane $(-1, 1)$. The distance to the origin is $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. The point is in quadrant II, so the angle $\theta = 3\pi/4$.

   (d) If A is similar to B, then prove that A^2 is similar to B^2.
   SOLUTION: If A is similar to B, then there is an invertible matrix P so that $A = PBP^{-1}$. If we then square A,
   $$A^2 = PBP^{-1}PBP^1 = PBBP^{-1} = PB^2P^{-1}$$
   Therefore, A^2 is similar to B^2.

9. Show that $\mathcal{B} = \left\{1, 2t, -2 + 4t^2\right\}$ is a basis for P_2. Be explicit about your reasoning!
   SOLUTION: We will use the fact that $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$, and the coordinate vectors of the $\mathcal{B}$ form a basis for $\mathbb{R}^3$.

   NOTE: You MUST set out this reasoning before you go into matrix-vector form!

   Using the coordinate mapping, we get the matrix
   $$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$
   These columns are clearly independent (three pivots, or non-zero determinant) and span $\mathbb{R}^3$, so therefore the corresponding polynomials in $\mathcal{B}$ are linearly independent and span $\mathbb{P}_2$.

10. Assume the mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by the following is linear.
    $$T(a_0 + a_1t + a_2t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$$
    Find the matrix representation of T relative to the standard basis $\mathcal{B} = \left\{1, t, t^2\right\}$
    SOLUTION: Notice that we can write, in general:
    $$\mathbf{p}(t) = c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3$$
    We could apply T to both sides and use the fact that T is linear to write the right side:
    $$T(\mathbf{p}(t)) = c_1T(\mathbf{p}_1) + c_2T(\mathbf{p}_2) + c_3T(\mathbf{p}_3)$$
    then taking the coordinates with respect to $\mathbb{B}$ to both sides:
    $$[T(\mathbf{p}(t))]_B = c_1[T(\mathbf{p}_1)]_B + c_2[T(\mathbf{p}_2)]_B + c_3[T(\mathbf{p}_3)]_B = [[T(\mathbf{p}_1)]_B \;\; [T(\mathbf{p}_2)]_B \;\; [T(\mathbf{p}_3)]_B][\mathbf{p}(t)]_B$$

We see that we just need the coordinate of the image under T of the three standard basis vectors.

$$T(1) = 3 + 5t \to \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \qquad T(t) = -2t + 4t^2 \to \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \qquad T(t^2) = t^2 \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These are the three columns to our matrix $[T]_B$.

11. If each row of the matrix A sums to the same number r, and A is $n \times n$, then what must one eigenvalue of A be, and what eigenvector? (Hint: Is there a vector $\mathbf{v}$ so that $A\mathbf{v}$ is a vector of row sums?)

    SOLUTION: The eigenvector is $(1, 1, \dots, 1)$ and the eigenvalue is s, since A times a vector of ones results in summing the rows of A.

12. Show that the eigenvalues of A and A^T are the same.

    SOLUTION: Remember that for proofs about eigenvalues, decide between the three equations:

    $$A\mathbf{v} = \lambda\mathbf{v} \qquad (A - \lambda I)\mathbf{v} = 0 \qquad \det(A - \lambda I) = 0$$

    Since we know that the determinant doesn't change if we transpose the matrix, we'll use that:

    $$\det(A - \lambda I) = 0 \quad \Leftrightarrow \quad \det((A - \lambda I^T) = 0 \quad \Leftrightarrow \quad \det(A^T - \lambda I^T) = 0$$

    Because I is diagonal, $I = I^T$, so therefore we have:

    $$\det(A^T - \lambda I) = 0$$

13. Find the eigenvalues and bases for the eigenspaces for each matrix below. If the matrix is diagonalizable in either PDP^{-1} or PCP^{-1} form, given P and D or C. Otherwise, state that as well.

    (a) $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$

    SOLUTION: The characteristic equation is $\lambda^2 - 6\lambda + 10 = 0$. Complete the square

    $$\lambda^2 - 6\lambda + 9 = -1 \quad \Rightarrow \quad (\lambda - 3)^2 = -1 \quad \Rightarrow \quad \lambda = 3 \pm i$$

    For $\lambda = 3 - i$, an eigenvector is found below:

    $$A - \lambda I = \begin{bmatrix} 2 + i & -4 \\ 1 & -2 + i \end{bmatrix} \quad \Rightarrow \quad (2 + i)\mathbf{v}_1 - 5\mathbf{v}_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 5 \\ 2 + i \end{bmatrix}$$

    To form P, the columns are the real and imaginary parts of $\mathbf{v}$,

    $$P = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

    (b) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$

    SOLUTION: The characteristic equation is $\lambda^2 - 4 = 0$, so $\lambda = \pm 2$.
    For $\lambda = 2$,

    $$A - \lambda I = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

    For $\lambda = -2$,

    $$A - \lambda I = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

    Therefore, P and D are given by:

    $$P = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

(c) $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$

The characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$, so $\lambda = -2$ is a double root. Finding the eigenspace,

$$A - \lambda I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This matrix is defective since the algebraic multiplicity was 2 and the geometric multiplicity is 1. This matrix cannot be diagonalized.

(d) $A = \begin{bmatrix} 2 & 2 \\ -4 & 6 \end{bmatrix}$

SOLUTION: The characteristic equation is $\lambda^2 - 8\lambda + 20 = 0$. Complete the square

$$\lambda^2 - 8\lambda + 16 = -4 \quad \Rightarrow \quad (\lambda - 4)^2 = -4 \quad \Rightarrow \quad \lambda = 4 \pm 2i$$

For $\lambda = 4 - 2i$, an eigenvector is found below:

$$A - \lambda I = \begin{bmatrix} -2+2i & 2 \\ -4 & 2+2i \end{bmatrix} \quad \Rightarrow \quad (-2+2i)\mathbf{v}_1 + 2\mathbf{v}_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2-2i \end{bmatrix} \text{ or } \mathbf{v} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

To form P, the columns are the real and imaginary parts of $\mathbf{v}$,

$$P = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix}$$

14. If A is similar to B, show that they have the same eigenvalues. Do they also have the same eigenvectors?

SOLUTION: By definition, if A is similar to B, there is a matrix P such that

$$A = PBP^{-1}$$

We can take two approaches. One is using the determinant (as done in class), but here is an alternative solution:

If λ is an eigenvalue of A, then

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Rightarrow \quad PBP^{-1}\mathbf{x} = \lambda\mathbf{x} \quad \Rightarrow \quad B(P^{-1}\mathbf{x}) = \lambda(P^{-1}\mathbf{x}) \quad \Rightarrow \quad B\mathbf{u} = \lambda\mathbf{u}$$

Therefore, λ is also an eigenvalue of B, but with a different eigenvector.

15. Prove that, if $n \times n$ matrix A is not invertible, then $\lambda = 0$ is an eigenvalue.

SOLUTION: Use the determinant. If A is not invertible, then

$$\det(A) = 0 \quad \Rightarrow \quad \det(A - 0I) = 0$$

so that $\lambda = 0$ is an eigenvalue.

16. Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.

SOLUTION: When we compute $A - \lambda I$, we only change the diagonal elements of A, so if A is triangular, then so is $A - \lambda I$.

We also know that the determinant of a triangular matrix is the product of the diagonal elements, so that

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$$

Therefore, $\lambda = a_{11}, a_{22}, \ldots, a_{nn}$, which are the diagonal elements of A.

17. Let $\mathcal{B}, \mathcal{C}$ be two sets of p vectors that represent bases for vector space V. Explain how we find $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Hint: You might start with a vector $\mathbf{x} \in V$ and expand it relative to $\mathcal{B}$.

SOLUTION: Starting with $\mathbf{x} \in V$, we can write:

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p$$

Now take the coordinate mapping of both sides with respect to basis $\mathcal{C}$:

$$[\mathbf{x}]_C = c_1[\mathbf{b}_1]_C + c_2[\mathbf{b}_2]_C + \cdots + c_p[\mathbf{b}_p]_C = [[\mathbf{b}_1]_C \;\; [\mathbf{b}_2]_C \;\; \cdots \;\; [\mathbf{b}_p]_C][\mathbf{x}]_B$$

So we see that the matrix is formed by taking the coordinates of the original basis with respect to the new basis:

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = [[\mathbf{b}_1]_C \;\; [\mathbf{b}_2]_C \;\; \cdots \;\; [\mathbf{b}_p]_C]$$

18. True or False? If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

(a) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
SOLUTION: False. The eigenvectors could be associated with a single eigenvalue (so that the geo. mult is 2 or more). Also note that, if we do have distinct eigenvalues, THEN it is true that the corresponding eigenvectors are linearly independent.

(b) If A is invertible, then A is diagonalizable.
SOLUTION: False. A matrix can be invertible, but not diagonalizable, like $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. A matrix can also be diagonalizable, but not invertible, like (in an extreme case) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) A is diagonalizable if $A = PDP^{-1}$ for some matrix D and invertible matrix P.
SOLUTION: False. Not just "some matrix D", but D must be diagonal!

(d) A is diagonalizable if A has n eigenvalues, counting multiplicities.
SOLUTION: False. If A has n distinct eigenvalues, then we can say that A is diagonalizable. The problem is that we don't know the geometric multiplicities of the eigenvalues.

(e) If A, B are similar, they have the same rank.
(I won't ask about this one. It's not something we've proven, but it is true. To prove it requires a bit more than I'll ask on an exam.) If you're wondering how to show it, you would show that the null spaces of A, B have the same dimension- And are in fact the same space.

(f) If A, B are row equivalent, then they have the same row space.
SOLUTION: If A, B are row equivalent, then the rows of A can be manipulated into the rows of B by using row operations. Both can therefore be changed to the same RREF, so they do share the same row space.

(g) If $\mathbf{u}, \mathbf{v}$ are eigenvectors of A, so is $\mathbf{u} + \mathbf{v}$.
SOLUTION: This is false in general, but if $\mathbf{u}$ and $\mathbf{v}$ are in the same eigenspace, then the statement is true.

(h) The rank of a matrix is equal to the number of nonzero rows.
SOLUTION: This is false in general. The rank is the dimension of the column space, which is the number of pivot columns (or pivot rows). If this had said "matrix in RREF", the statement would have been true.

19. Show that the rank of AB is less than or equal to the rank of A. Hint: Think about the columns of AB.

SOLUTION: Remember that the columns of AB are linear combinations of the columns of A. Therefore, the number of linearly independent columns cannot increase (but it can decrease). Therefore, the rank of AB is less than or equal to the rank of A.

20. Suppose $A = PDP^{-1}$ with a suitable 2×2 matrix P and $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

(a) If $B = 3I - 2A + A^2$, show that B is diagonalizable by finding an appropriate factorization of B.
SOLUTION: Substitute PDP^{-1} for A, PP^{-1} for I, and PD^2P^{-1} for A^2:

$$B = 3PP^{-1} - 2PDP^{-1} + PD^2P^{-1} = P(3I - 2D + D^2)P^{-1}$$

where

$$D^2 - 2D + 3I = \begin{bmatrix} 2^2 - 2(2) + 3 & 0 \\ 0 & 7^2 - 2(7) + 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 38 \end{bmatrix}$$

(b) From your previous answer, if λ is an eigenvalue of A, then what would an eigenvalue of $A^2+bA+cI$ be?
SOLUTION: $\lambda^2 + b\lambda + c$.

21. Whoops! This is the same as question 17. Sorry about the duplication!

22. In $\mathbb{P}_2$, let $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\}$. Let $\mathcal{C} = \{1, t, t^2\}$.

(a) Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.
SOLUTION: Recall that in general, if $\mathcal{B}$ has basis vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, then

$$\mathbf{x} = c_1\mathbf{p}_1(t) + c_2\mathbf{p}_2 + c_2\mathbf{p}_3$$

Taking the coordinates with respect to $\mathcal{C}$ will give us:

$$[\mathbf{x}]_C = c_1[\mathbf{p}_1(t)]_C + c_2[\mathbf{p}_2]_C + c_2[\mathbf{p}_3]_C = [[\mathbf{p}_1(t)]_C \;\; [\mathbf{p}_2]_C \;\; [\mathbf{p}_3]_C][\mathbf{x}]_B$$

So therefore, the change of coordinates matrix is simply the coordinates of each basis vector with respect to the standard basis:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

(b) Find the coordinates of t^2 with respect to the basis $\mathcal{B}$.
SOLUTION: Invert the matrix, or solve the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

23. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis for some vector space V. Let $T : \mathbb{R}^3 \to V$ be a linear transformation:

$$T(x_1, x_2, x_3) = (2x_3 - x_2)\mathbf{c}_1 - (2x_2)\mathbf{c}_2 + (x_1 + 3x_3)\mathbf{c}_3$$

Find the matrix for T relative to $\mathcal{E}$ and $\mathcal{C}$.

SOLUTION: This is very much like what we've done before. We start with a vector in $\mathbb{R}^3$, then we apply T, then we'll apply the coordinate mapping:

$$\mathbf{x} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$$

$$T(\mathbf{x}) = \alpha_1 T(\mathbf{e}_1) + \alpha_2 T(\mathbf{e}_2) + \alpha_3 T(\mathbf{e}_3)$$

$$[T(\mathbf{x})]_C = \alpha_1[T(\mathbf{e}_1)]_C + \alpha_2[T(\mathbf{e}_2)]_C + \alpha_3[T(\mathbf{e}_3)]_C = [[T(\mathbf{e}_1)]_C \;\; [T(\mathbf{e}_2)]_C \;\; [T(\mathbf{e}_3)]_C][\mathbf{x}]_E$$

So we're looking for the coordinates of T applied to the standard basis vectors, in terms of $\mathcal{C}$.

- $T(1,0,0) = \mathbf{c}_3 \Rightarrow (0,0,1)$
- $T(0,1,0) = -\mathbf{c}_1 - 2\mathbf{c}_2 \Rightarrow (-1,-2,0)$
- $T(0,0,1) = 2\mathbf{c}_1 + 3\mathbf{c}_3 \Rightarrow (2,0,3)$

These are the columns of the matrix.

24. Find $T(a_0 + a_1 t + a_2 t^2)$ if T is the linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$ whose matrix relative to $\mathcal{B} = \{1, t, t^2\}$ is given by:

$$[T]_B = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

SOLUTION: Recall that $[T(\mathbf{p})]_B = [T]_B[\mathbf{p}]_B$, so we just multiply the matrix times (a_0, a_1, a_2) to get

$$\begin{bmatrix} 3a_0 + 4a_1 \\ 5a_1 - a_2 \\ a_0 - 2a_1 + 7a_2 \end{bmatrix} \to T(\mathbf{p}) = (3a_0 + 4a_1) + (5a_1 - a_2)t + (a_0 - 2a_1 + 7a_2)t^2$$

25. Suppose in Matlab, I have the matrix A below, and the program gave the following matrices for V and D. Find P, C so that $A = PCP^{-1}$.

$$A = \begin{bmatrix} 3 & 2 & 3 & 1 \\ 1 & 3 & 3 & 3 \\ 3 & 1 & 3 & 3 \\ 3 & 2 & 2 & 3 \end{bmatrix}$$

```
>> [V,D]=eig(A)
V =
-0.4  -0.1 + 0.4i  -0.1 - 0.4i  -0.3
-0.5  -0.5 + 0.0i  -0.5 - 0.0i   0.7
-0.5   0.1 - 0.4i   0.1 + 0.4i  -0.4
-0.5   0.4 + 0.0i   0.4 - 0.0i   0.2

D =
9.7   0            0            0
0     0.2 + 1.3i   0            0
0     0            0.2 - 1.3i   0
0     0            0            1.7
```

SOLUTION:

$$P = \begin{bmatrix} -0.4 & -0.1 & 0.4 & -0.3 \\ -0.5 & -0.5 & 0.0 & 0.7 \\ -0.5 & 0.1 & 0.4 & 0.4 \\ -0.5 & 0.4 & 0.0 & 0.2 \end{bmatrix} \qquad C = \begin{bmatrix} 9.7 & 0 & 0 & 0 \\ 0 & 0.2 & 1.3 & 0 \\ 0 & -1.3 & 0.2 & 0 \\ 0 & 0 & 0 & 1.7 \end{bmatrix}$$

26. If we think about a block form of a matrix, if we have

$$G = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$$

where A, B are square (not necessarily the same size), then $\det(G) = (\det A)(\det(B))$. Use this to help find the eigenvalues of the matrices below:

$$G_1 = \begin{bmatrix} 3 & -2 & 8 \\ 0 & 3 & -2 \\ 0 & 2 & 3 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 5 & -6 & -7 \\ 2 & 4 & 5 & 2 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

SOLUTION: For G_1, the eigenvalues are $\lambda = 3$, and from the 2×2 block, $\lambda = 3 \pm 2i$.

For G_2, we have two 2×2 blocks, but these blocks aren't in the right form to read off the eigenvalues. For the top block,

$$\lambda^2 - 5\lambda - 6 = 0 \quad \Rightarrow \quad (\lambda + 1)(\lambda - 6) = 0 \quad \Rightarrow \quad \lambda = -1, 6$$

And for the second block,

$$\lambda^2 + 6\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda + 1)(\lambda + 5) = 0 \quad \Rightarrow \quad \lambda = -1, -5$$

And these are the four eigenvalues.