Review Material, Exam 3

The exam will cover the material from Sections 4.1-4.7 and 5.1-5.5. In these sections, we've looked at vector spaces, subspaces, the four fundamental subspaces, coordinate systems and coordinate mapping, the dimension and rank, and the change of basis. In chapter 5, we have studied eigenvalues and eigenspaces.

Vector Spaces, (4.1-4.7)

1. You don't need to memorize the 10 axioms on page 217.
2. Be familiar with some template vector spaces: $\mathbb{R}^n$, $\mathbb{P}_n$, $\mathbb{P}$, $C[a,b]$, $M_{m\times n}$
3. Know these definitions: A subspace, a linearly independent set, a basis, the coordinates of $\mathbf{x}$ (with respect to a given basis), the dimension of a subspace, an isomorphism, the rank of a matrix. The four fundamental subspaces associated with a matrix A (be able to define each one), the kernel of a transformation, the change of coordinates matrix. A steady state vector for a Markov Chain. A regular stochastic matrix.
4. Theorems for computation: 1, 4, 5, 6, 7, 9, 10, 11, 13, 15. (and the equation at the bottom of p. 241)

   These are theorems that you should know for computational purposes (you might think of them as "basic facts").
5. Theorems to know: 2 (Null space is a subspace), 3 (Col space is a subspace), 8 (Isomorphism for Isomorphic Spaces), 12 (The basis theorem), 14 (The rank theorem).
6. Skills:
   - Prove that a given set is or is not a subspace.
   - Given a matrix A, be able to compute a basis for the column space, the null space and the row space (not the null space of A^T).
   - Find the kernel of a given transformation and describe the range of the transformation.
   - Understand how row operations effect the the 4 fundamental subspaces (for example, the subspaces for a matrix A versus its RREF, B).
     - Row operations do not effect the relationship among the columns of A, but they do effect the column spaces (the column spaces of A, B may not be the same).
     - Row operations do effect the relationship among the rows of A, but the row spaces of A, B are the same.
     - Row operations do not effect the set of solutions to $A\mathbf{x} = \mathbf{0}$, so the null spaces of A, B are the same.

- Find the coordinates of a vector given a basis (both in $\mathbb{R}^n$ using the change of coordinates matrix, and for vector spaces that are not $\mathbb{R}^n$, like $\mathbb{P}_n$.
- Be able to compute the dimension of a vector space.
- Be able to compute the rank of a matrix. Use that to compute the dimensions of the four fundamental subspaces.
- Understand what it means to say that two vectors spaces are **isomorphic**.
- Theorem: Any finite dimensional vector space V (with dimension n) is isomorphic to $\mathbb{R}^n$ using the coordinate mapping as the isomorphism.

7. Given bases $\mathcal{B}$ and $\mathcal{C}$, find a formula that will, given the coordinates in $\mathcal{B}$, give you the coordinates with respect to $\mathcal{C}$.

5.1-5.5

Key definitions:

Eigenvalue, eigenvector, eigenspace, characteristic polynomial (and equation), algebraic and geometric multiplicity of an eigenvalue, defective matrix. A is similar to B, diagonalizable.

Complex conjugate, polar form of a complex number, magnitude (or length) for a complex number, argument for a complex number. Eulers Formula (with the complex exponential). Form of "diagonalization" when the eigenvalues are complex.

Important Theorems:

For these theorems, be able to use them. I won't ask you to state them verbatim, but be able to use them.

- Theorems 1, 2, 5, 7, 8 (really Equations 3, 4 on p. 289)

We learned some theorems earlier in chapter 3 (determinants). We may need to use these properties in proofs about evals.

Be able to prove these:

- Theorem 4 (similar matrices have same evals)
- Theorem 6 (n distinct evals implies diagonalizable)

Relationship of the eigenvalues (or diagonalizability) to the invertibility of a matrix.

Chapter 5 skills

- Be able to compute basic operations using complex numbers (in particular, multiplication and division). Be able to convert between complex numbers and the polar form using Eulers Formula (not in text). Use the handout from class as your guide.
- Know the three equations associated with eigenvalues and eigenvectors:

$$A\mathbf{x} = \lambda\mathbf{x} \qquad (A - \lambda I)\mathbf{x} = \mathbf{0} \qquad \det(A - \lambda I) = 0$$

  and know which one to use (either to compute something or to prove something).
- Find the characteristic equation and eigenvalues of a 2×2 matrix. Find the eigenvalues of a triangular matrix. Find a basis for an eigenspace. (NOTE: For matrices larger than 2×2, the eigenvalues would either be given, the matrix would have a special form, or just compute the characteristic polynomial).
- Be able to factor a matrix (PCP^{-1}) if it has complex eigenvalues (Theorem 9, p. 340)
- If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, explain the "action" of C using scaling r and CCW rotation by angle θ.
- Given A (bigger than 2×2), be able to interpret the action of a block diagonal matrix.