Review Material, After the Third Exam

The material after the third exam includes most of Chapter 6, 6.1-6.6.

Important Definitions

Inner product (dot product), norm (length), distance, angle all in $\mathbb{R}^n$. Orthogonal set, orthogonal projection (also see below). Gram-Schmidt process. Orthogonal matrix, least squares solution of $A\mathbf{x} = \mathbf{b}$, normal equation.

Skills (Partial List)

Be able to compute things like dot product, angle between vectors, distance between vectors.

Projections, projections, and projections!

- Compute the projection of $\mathbf{y}$ onto $\mathbf{u}$.
- Compute the projection of $\mathbf{y}$ onto subspace W given an orthogonal basis for W.
- Decompose a vector using the Orthogonal Decomposition Theorem, and a subspace W.
- Given a set of vectors, find an orthonormal set of vectors that are "equivalent" in the sense of Gram-Schmidt. (Tell me what that means...)

Compute the least squares solution using the normal equations.

Given a model function (like in 6.6) and some data, be able to set up the matrix equation.

Computational Theorems

These are theorems you should review, but only for the properties or the result. You won't be asked to state these, but you should be able to use them.
Theorem 1 (properties of dot), 4 (orthog implies lin ind), 5 (formula for coordinates), 6 (check to see if you have o.n. cols), 9 (The best approximation theorem), 10 (formula for matrix version of the projection), 13 (Least Squares and the Normal Equation), 14 (Solving the least squares when full rank).

Theorems you should be able to prove

Theorem 2 (Pythagorean Theorem), 3 (Row space is orthogonal to the null space), 7 (Properties of a matrix with o.n. columns).

Important Theorems to Know

Theorem 8: The Orthogonal Decomposition Theorem. Theorem 11: The Gram-Schmidt process.

Review Questions

1. Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, given A and $\mathbf{b}$ below. Note that the columns of A are orthogonal, and use that fact.

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

2. Find the line that best fits the data: $(-1, -1), (0, 2), (1, 4), (2, 5)$. Do this by first finding a matrix equation that you will then find the least squares solution to (by using the normal equations).

3. Show that if $\mathbf{x} \in \text{Null}(A)$, then $\mathbf{x} \in \text{Null}(A^T A)$.

   Show that if $A^T A\mathbf{x} = 0$, then $\|A\mathbf{x}\| = ?$.

   Use the above to show that, if $\mathbf{x} \in \text{Null}(A^T A)$, then $\mathbf{x} \in \text{Null}(A)$.

   Altogether, this problem is showing that the null spaces of A and $A^T A$ are the same!

4. Using the last problem, what can we conclude about the rank of A versus the rank of $A^T A$?

5. Suppose I have a model equation: $y = \beta_0 + \beta_1 \sin(v) + \beta_2 \ln(w)$.

   Given the following data, set up the matrix equation from which we could determine a least squares solution for the β's:

v	w	y
-1	2	1
1	1	2
0	3	-1
3	2	0

   (Do NOT actually solve for the β's, just set up the matrix equation).

6. Given vectors $\mathbf{u}, \mathbf{v}$ in the vector space $\mathbb{R}^n$ with the usual dot product as inner product, show that the Pythagorean Theorem still holds. That is, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other, then:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

   (Note: This question is asking you to prove Theorem 2)

7. True or False, and explain: For every non-zero vector $\mathbf{v} \in \mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

8. Let A be a 6×4 matrix with orthonormal columns. Make appropriate calculations to show that $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$.

9. Let $\mathbf{x} = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, and let $W = \text{span}(\mathbf{u}, \mathbf{v})$. Decompose $\mathbf{x}$ into a sum of vectors- one in W, and one in $W^\perp$.

10. Suppose an experiment produces (x, y) data: $(2, 5), (3, 6), (4, 8), (5, 10)$, and a scientist wants to model that data with an equation of the form $y = \beta_1 x + \beta_2 x^2 + \beta_3 \mathrm{e}^{-x}$. Write the design matrix, the unknown parameter vector and the observation vector for this problem (with the entries filled in). Do NOT solve for the unknown parameters.

11. The given set of vectors is a basis for subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W:

$$\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

12. In the following, let $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$, and find the vector in W that is closest to $\mathbf{z}$.

$$\mathbf{z} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$