Solutions to the Review 4 Exercises

1. Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, given A and $\mathbf{b}$ below. Note that the columns of A are orthogonal, and use that fact.

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

SOLUTION: Since the columns of A are orthogonal, we can compute the $\hat{\mathbf{b}}$ directly.

$$\hat{\mathbf{b}} = \frac{\mathbf{b}^T\mathbf{a}_1}{\mathbf{a}_1^T\mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{b}^T\mathbf{a}_2}{\mathbf{a}_2^T\mathbf{a}_2}\mathbf{a}_2 = \frac{7}{9}\mathbf{a}_1 + \frac{1}{9}\mathbf{a}_2 = A\hat{\mathbf{x}}$$

so we can read $\hat{\mathbf{x}}$ off: $[7/9, 1/9]^T$. (See page 414 for another example).

2. Find the line that best fits the data: $(-1,-1), (0,2), (1,4), (2,5)$. Do this by first finding a matrix equation that you will then find the least squares solution to (by using the normal equations).

SOLUTION: The model equation is $y = \beta_0 + \beta_1 x$, so the matrix equation is:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

Forming the normal equations, we have:

$$A^T A\vec{\beta} = A^T\mathbf{y} \quad \Rightarrow \quad \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 2 \end{bmatrix}$$

3. Show that if $\mathbf{x} \in \text{Null}(A)$, then $\mathbf{x} \in \text{Null}(A^T A)$.

SOLUTION: If $\mathbf{x} \in \text{Null}(A)$, then $A\mathbf{x} = \mathbf{0}$. Multiplying both sides by A^T, we see that $A^T A\mathbf{x} = \mathbf{0}$, so that $\mathbf{x} \in \text{Null}(A^T A)$.

Show that if $A^T A\mathbf{x} = 0$, then $\|A\mathbf{x}\| =?$.

SOLUTION: Looking at the expression to the left, it is similar to what we have if we compute $\|A\mathbf{x}\|$. In fact:

$$\|A\mathbf{x}\|^2 = (A\mathbf{x}) \cdot (A\mathbf{x}) = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x}$$

Now, if $A^T A\mathbf{x} = \mathbf{0}$ then $\mathbf{x}^T A^T A\mathbf{x} = 0$ so that $\|A\mathbf{x}\|^2 = 0$.

Use the above to show that, if $\mathbf{x} \in \text{Null}(A^T A)$, then $\mathbf{x} \in \text{Null}(A)$.

SOLUTION: In the previous problem, we showed that if $\mathbf{x} \in \text{Null}(A^T A)$, then $\|A\mathbf{x}\| = 0$. This implies that $A\mathbf{x} = \mathbf{0}$, or equivalently, that $\mathbf{x} \in \text{Null}(A)$.

Altogether, this problem is showing that the null spaces of A and $A^T A$ are the same!

4. Using the last problem, what can we conclude about the rank of A versus the rank of $A^T A$?

   SOLUTION: If A is $m \times n$, then the null spaces of A and $A^T A$ are the same subspaces of $\mathbb{R}^n$- thus they also have the same dimension. Therefore, the dimension of $\text{Row}(A)$ and $\text{Row}(A^T A)$ are the same, and therefore, the dimension of $\text{Col}(A)$ and $\text{Col}(A^T A)$ are the same. Therefore, A and $A^T A$ have the same rank.

5. Suppose I have a model equation: $y = \beta_0 + \beta_1 \sin(v) + \beta_2 \ln(w)$.

   Given the following data, set up the matrix equation from which we could determine a least squares solution for the β's:

$$\begin{array}{ccc} v & w & y \\ -1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 3 & -1 \\ 3 & 2 & 0 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & \sin(-1) & \ln(2) \\ 1 & \sin(1) & \ln(1) \\ 1 & \sin(0) & \ln(3) \\ 1 & \sin(3) & \ln(2) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

   Side Remark: In Matlab, you could solve this:

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v=[-1 1 0 3]'; w=[2 1 3 2]'; y=[1 2 -1 0]';
A=[ones(4,1), sin(v), log(w)];
beta=inv(A'*A)*A'*y;
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6. Given vectors $\mathbf{u}, \mathbf{v}$ in the vector space $\mathbb{R}^n$ with the usual dot product as inner product, show that the Pythagorean Theorem still holds. That is, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other, then:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

   SOLUTION: Write out the left side in terms of the dot product, and expand.

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}$$

   Since $\mathbf{u} \cdot \mathbf{v} = 0$, this expression reduces to

$$\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

7. True or False, and explain: For every non-zero vector $\mathbf{v} \in \mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

SOLUTION: False, unless $\mathbf{v}$ is unit length. Then

$$\text{Proj}_{\mathbf{v}}(\mathbf{x}) = \mathbf{v}\left(\frac{\mathbf{v}^T\mathbf{x}}{\mathbf{v}^T\mathbf{v}}\right) = (\mathbf{v}\mathbf{v}^T)\mathbf{x}$$

The last equality holds if $\|\mathbf{v}\| = 1$.

8. Let A be a 6×4 matrix with orthonormal columns. Make appropriate calculations to show that $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$.

SOLUTION:

$$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x}^T A^T A\mathbf{y}$$

If A has o.n. cols, then $A^T A$ is the 4×4 identity matrix.

$$\mathbf{x}^T A^T A\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$$

9. Let $\mathbf{x} = \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, and let $W = \text{span}(\mathbf{u}, \mathbf{v})$. Decompose $\mathbf{x}$ into a sum of vectors- one in W, and one in $W^\perp$.

SOLUTION: This is the computational version of the orthogonal decomposition theorem- You'll note that $\mathbf{u}, \mathbf{v}$ are orthogonal vectors! We take

$$\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$$

where $\hat{\mathbf{x}} \in W$ (the orthogonal projection) and $\mathbf{z} = \mathbf{x} - \hat{\mathbf{x}}$, which is in $W^\perp$.

$$\hat{\mathbf{x}} = \frac{0 - 6 + 4}{2 + 1 + 1}\mathbf{u} + \frac{0 + 12 + 16}{1 + 4 + 16}\mathbf{v} = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

As a quick double check, look to see if your two vectors are orthogonal!

10. Suppose an experiment produces (x, y) data: $(2, 5), (3, 6), (4, 8), (5, 10)$, and a scientist wants to model that data with an equation of the form $y = \beta_1 x + \beta_2 x^2 + \beta_3 e^{-x}$. Write the design matrix, the unknown parameter vector and the observation vector for this problem (with the entries filled in). Do NOT solve for the unknown parameters.

11. The given set of vectors is a basis for subspace W. Use the Gram-Schmidt process to produce an orthogonal basis for W:

$$\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

(Its OK if you do not normalize them since we're doing these by hand.)

12. In the following, let $W = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$, and find the vector in W that is closest to $\mathbf{z}$.

$$\mathbf{z} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

SOLUTION: We should find that projecting $\mathbf{z}$ into W yields the following vector, which represents the vector in W closest to $\mathbf{z}$.

$$\hat{\mathbf{z}} = \frac{2}{3}\mathbf{v}_1 - \frac{7}{3}\mathbf{v}_2 = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}$$

It wasn't asked, but the distance between $\mathbf{z}$ and the plane that is W is $\|\mathbf{z} - \hat{\mathbf{z}}\| = \sqrt{4^2 + 4^2 + 4^2} = \sqrt{48}$