## Example Questions, Exam 3, Math 240

1. Let A be given as below, and B is row equivalent:

$$A = \begin{bmatrix} -1 & -5 & 3 & 9 \\ -48 & -40 & 24 & 92 \\ 94 & 70 & -42 & -166 \\ -48 & -40 & 24 & 92 \end{bmatrix} \qquad \qquad B = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 10 & -3 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We also note two facts:  $\lambda = 4$  is an eigenvalue of A, and  $\mathbf{u} = [1, 0, 2, 0]^T$  is an eigenvector of A (not necessarily with  $\lambda = 4$ ).

- (a) Find a basis for the eigenspace  $E_4$ :
- (b) What is the eigenvalue for the eigenvector **u**?
- (c) You might have noticed that the second and fourth rows are the same. Does that imply we have a certain eigenvalue? Find a basis for its eigenspace. To save you some time, we have included B which is row equivalent to A.
- (d) What is the characteristic polynomial of A?
- (e) Show that A is diagonalizable by finding an appropriate P and D.
- 2. Short Answer:
  - (a) Show that if  $A^2$  is the zero matrix, the only eigenvalue of A is zero.
  - (b) Write the complex number in a + ib form:  $\frac{1-3i}{2+i}$ .
  - (c) Write the complex number in polar form,  $re^{i\theta}$ : -1 3i
  - (d) Normalize the vector  $[1, -2, 1, 1]^T$ .
  - (e) Suppose A is 3 × 3, and u is an eigenvector of A corresponding to an eigenvalue of 7.
    Is u an eigenvector of 2I − A? If so, find the corresponding eigenvalue. If not, explain why not.
  - (f) True or False? A matrix with orthonormal columns is an orthogonal matrix.
- 3. Show the following: If U, V are orthogonal matrices, then so is UV.
- 4. Let  $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$ , diagonalize the matrix A.
- 5. If each row of the matrix A sums to the same number r, and A is  $n \times n$ , then what must one eigenvalue of A be, and what eigenvector? (Hint: Is there a vector **v** so that A**v** is a vector of row sums?)
- 6. Let U be  $m \times n$  with orthonormal columns. Show that  $||U\mathbf{x}|| = ||\mathbf{x}||$ .

Let U be  $m \times n$  with orthonormal columns. Show that the angle between **x** and **y** is the same as the angle between U**x** and U**y**.

- 7. Show that the eigenvalues of A and  $A^T$  are the same.
- 8. Show that if  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are eigenvectors corresponding to distinct eigenvalues, then the vectors are linearly independent.
- 9. Find the eigenvalues and bases for the eigenspaces if  $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$ .
- 10. Compute an appropriate factorization for the matrix  $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$ .
- 11. Let matrix A be  $m \times n$ . Show that the row space is orthogonal to the null space.
- 12. If  $\mathbf{u} = [3, 2, -5, 0]$  and  $\mathbf{v} = [1, 1, -1, 2]$ , then compute:

- (a) The distance between  $\mathbf{u}$  and  $\mathbf{v}$ .
- (b) The angle between  $\mathbf{u}$  and  $\mathbf{v}$  (leave in exact form).
- (c) The orthogonal projection of  $\mathbf{u}$  onto  $\mathbf{v}$
- (d) Having an orthogonal projection means what two vectors are orthogonal? Show that this is indeed the case.
- 13. Prove the Pythagorean Theorem for two vectors  $\mathbf{x}$  and  $\mathbf{y}$ . If  $\mathbf{x}$  and  $\mathbf{y}$  are orthogonal vectors in  $\mathbb{R}^n$ , then

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

- 14. If A is similar to B, show that they have the same eigenvalues. Do they also have the same eigenvectors?
- 15. Prove that if the set  $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$  form a basis for subspace W, and  $\mathbf{x}$  is orthogonal to each  $\mathbf{v}_i$ , for i = 1 to k, then  $\mathbf{x}$  is orthogonal to W. (Hint: Start with a generic vector  $\mathbf{w} \in W$ , and show that  $\mathbf{x} \cdot \mathbf{w} = 0$ .)
- 16. Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.
- 17. (More) True or False? If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.
  - (a) If  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
  - (b) If A is invertible, then A is diagonalizable.
  - (c) The orthogonal projection of **y** onto a vector **v** is the same as the orthogonal projection of **y** onto  $c\mathbf{v}$  whenever  $c \neq 0$ .
  - (d) If A is an orthogonal matrix, then  $A^T$  is an orthogonal matrix.
  - (e) If A, B have the same eigenvalues, then they are similar.
- 18. Show (by using the projection formula for one vector onto another) that if U is a matrix with orthonormal columns, then  $UU^T \mathbf{x}$  is the projection of  $\mathbf{x}$  onto the columns of U.

(NOTE: This formula also tells us that the coordinates of  $\mathbf{x}$  with respect to the columns of U is given by  $U^T \mathbf{x}$ ).

Similarly, show that, if U has orthonormal columns, then  $U^T U = I$ .

- 19. When we originally partitioned  $\mathbb{R}^n$  into one part as the row space and the other part as the null space for a matrix, we said that the only thing in both spaces was the zero vector. How do we know that there is not a vector in  $\mathbb{R}^n$  that is in neither the row space nor the null space? (Think about 6.3- the orthogonal decomposition theorem, for example).
- 20. Use Gram-Schmidt to give us an orthogonal set of vectors to replace the columns of the matrix A below.

$$\left[\begin{array}{rrrrr} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

21. Let A = QR.

- (a) If A is  $4 \times 3$ , what dimensions could Q and R be, if we use Gram-Schmidt to compute Q?
- (b) Here are matrices A, Q and R. See if you can tell which is which.

$$\begin{bmatrix} 1 & -3 & 1/2 \\ 0 & -1 & -5/2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \begin{bmatrix} -1 & -3 & 1 \\ 3 & -1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

- (c) If A = QR, as given in the previous part, then write  $\mathbf{a}_3$  as a linear combination of  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ .
- (d) If  $A\mathbf{x} = \mathbf{b}$ , and A = QR, then write the solution  $\mathbf{x}$  in terms of Q, R.
- (e) Suppose  $A = PDP^{-1}$  with a suitable  $2 \times 2$  matrix P and  $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$ .
  - i. If  $B = 3I 2A + A^2$ , show that B is diagonalizable by finding an appropriate factorization of B.
  - ii. From your previous answer, if  $\lambda$  is an eigenvalue of A, then what would an eigenvalue of  $A^2 + bA + cI$  be?

## Additional Questions, 4.9- Markov Chains

1. (A calculator may be used on this problem. On the exam, I will choose numerical values that will be easy to work with, since you won't have a calculator).

On any given day, a student is either healthy or ill. Of the students that are healthy today, 95% will be healthy tomorrow. Of the students that are ill today, 55% will be ill tomorrow.

- (a) What is the stochastic matrix for this system if we model it as a Markov chain?
- (b) Suppose 20% of the students are ill on Monday. What is the fraction or percentage of the students that will likely be ill on Tuesday?
- (c) What happens to the percentages in the long run?
- 2. Show that every  $2 \times 2$  stochastic matrix has at least one steady-state vector.

Hint: Any such matrix can be written in the form  $P = \begin{bmatrix} 1 - \alpha & \beta \\ \alpha & 1 - \beta \end{bmatrix}$ , where  $\alpha, \beta$  are constants between 0 and 1.

Also, how many steady state vectors are there if  $\alpha = \beta = 0$ ?

3. Let S be the  $1 \times n$  row of all ones. If P is an  $n \times n$  square matrix with all non-negative entries, show that P is stochastic if SP = S.