Review Material, After the Third Exam

The material after the third exam includes: 6.5 (Least Squares), 6.6 (Applications to Linear Models), 6.7 (Inner Product Spaces), then we looked at 7.1 (Symmetric Matrices and the Spectral Theorem), and finally we looked at 7.4 (The SVD).

Important Definitions

Orthogonal matrix, Least squares solution of $A\mathbf{x} = \mathbf{b}$, normal equation, symmetric matrix, orthogonally diagonalizable. Given a general inner product, be able to define the norm (or length), the distance and the angle for vectors in a general vector space. Define the singular values of a matrix, and the pseudoinverse (in terms of the SVD).

Skills (Partial List)

> Compute the least squares solution using the normal equations. Set up a matrix equation for a given linear model and find the least squares solution.
>
> Use Gram-Schmidt and be able to compute projections in a general vector space (using a given inner product).
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> Orthogonally diagonalize a symmetric matrix. Be able to project a vector into a subspace spanned by a given set of orthonormal vectors (and be able to use matrix notation). Understand how the Spectral Decomposition works (decomposition into rank one matrices).
>
> Be able to compute the SVD by hand for small matrices. Understand the specific relationship between the SVD for a matrix A and the four fundamental subspaces.
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> For the **take home portion**, be able to compute the SVD of a matrix using a computer program (Matlab or Octave), and be able to determine the rank. Be able to compute the pseudoinverse and solve a least squares problem, or be able to project data into the space spanned by either the columns of U or V. A good indicator of the type of problem was distributed on our last day of class. All the solutions to those problems are on our class website.

Important Theorems

6.5.14 (Solving Normal Eqns), 7.1.2 (Symmetric is equivalent to orth. diag.), 7.1.3 (The Spectral Theorem), 7.4.10 (The SVD).

Review Questions

1. Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, given A and $\mathbf{b}$ below. Note that the columns of A are orthogonal, and use that fact to directly compute $\hat{\mathbf{b}}$, and along the way.

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

2. Find the line that best fits the data: $(-1, -1), (0, 2), (1, 4), (2, 5)$. Do this by first finding a matrix equation that you will then find the least squares solution to (by using the normal equations).

3. Suppose A is $m \times n$ with linearly independent columns and $\mathbf{b}$ is in $\mathbb{R}^m$. Use the normal equations to produce a formula for $\hat{\mathbf{b}}$, the projection of $\mathbf{b}$ onto the column space of A. (Hint: First find $\hat{\mathbf{x}}$ which does not require an orthogonal basis for $\text{Col}(A)$.)

4. Show that $\text{Null}(A) = \text{Null}(A^T A)$ by taking the following steps:

   (a) Show that if $A^T A\mathbf{x} = 0$, then $\|A\mathbf{x}\| = ?$.
   Use the above to conclude that, if $\mathbf{x} \in \text{Null}(A^T A)$, then $\mathbf{x} \in \text{Null}(A)$.

   (b) The other direction is easy- Show that if $\mathbf{x} \in \text{Null}(A)$, then it is in $\text{Null}(A^T A)$.

5. Continuing with the previous question, what can we conclude about the rank of A vs. the rank of $A^T A$?

6. Suppose I have a model equation: $y = \beta_0 + \beta_1 \sin(v) + \beta_2 \ln(w)$.

   Given the following data, set up the matrix equation from which we could determine a least squares solution for the β's:

v	w	y
-1	2	1
1	1	2
0	3	-1
3	2	0

   (Do NOT actually solve for the β's, just set up the matrix equation).

7. Given vectors $\mathbf{u}, \mathbf{v}$ in the vector space $\mathbb{R}^n$ with the usual dot product as inner product, show that the Pythagorean Theorem still holds. That is, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other, then:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

   Hint: Expand the norm using the dot product.

8. Orthogonally diagonalize the symmetric matrix $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$.

9. True or False, and explain: For every non-zero vector $\mathbf{v} \in \mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

10. Show that, if A is symmetric, then any two eigenvectors from distinct eigenvalues are orthogonal. Hint: Start with $\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2$, and see if you can transform this into $\lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2$.

11. Suppose we have the matrix $A = [1, 1, 1]$.
    (a) What will the singular values of A be? (Try to compute them in the easiest possible way).
    (b) Find (by hand) the reduced SVD for the matrix A. See if you can do it without any computation.
    (c) Find a basis for the null space of A using the rest of the SVD that hasn't been computed yet (this one we'll need to compute).

12. Show that the eigenvalues of $A^T A$ are non-negative. Hint: Consider $\|A\mathbf{v}_i\|^2$.

13. Suppose the SVD was given as the following:

$$A = \begin{bmatrix} 0.65 & -0.75 & 0 \\ 0 & 0 & 1 \\ 0.75 & 0.65 & 0 \end{bmatrix} \begin{bmatrix} 15.91 & 0 & 0 \\ 0 & 3.26 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.52 & -0.62 & -0.57 \\ -0.27 & 0.76 & -0.57 \\ -0.80 & 0.14 & 0.57 \end{bmatrix}^T$$

    (a) What is the rank of A?
    (b) Write a basis for the column space and null space of A.
    (c) Write the matrix product for the pseudoinverse of A (you don't need to multiply it out).

14. Suppose A is square and invertible. Find the SVD of A^{-1}.

15. Show that if A is square, then the absolute value of the determinant, $|\det(A)|$, is the product of the singular values of A.